# 4. HOLDING THIS, HOLDING THAT

*letters used to represent other objects: variables*

In the first few sections of this book, you've seen letters (like $x$ and $S$) being used to represent numbers and sets: letters used in this way are called *variables*. The author has tried *very hard* to minimize the appearance of variables in these first few sections: however, taking away a mathematician's ability to use variables is like asking to empty a swimming pool with a teaspoon. Maybe the job can be done, but it will take very long, and be very laborious. The purpose of this section is to explain why mathematicians are so fond of variables—indeed, why they *must* use variables—and to illustrate the power that the use of variables gives them.

*variables 'hold' objects*

Variables are used to 'hold' objects; we get to specify the objects to be 'held'. Here are a few examples:

- A rectangle can have any positive height. If the letter $h$ is used to represent this height, then $h$ can 'hold' any positive number.
- If the letter $b$ is used to represent the number of books in a home library, then $b$ can 'hold' the numbers $0$ , $1$ , $2$ , $\ldots$ .
- Suppose we let $S$ denote any subset of the real numbers. Then, $S$ might 'hold' the interval $(1,2]$ , or the set $\{1,2\}$ , or the set $\mathbb{Z}$ .

Therefore, a variable is a 'holder', with some 'supplier' lurking in the background, specifying what the variable gets to hold.

**EXERCISES**

1. On three separate number lines, illustrate the subsets of the real numbers that are mentioned above: $(1,2]$ , $\{1,2\}$ , and $\mathbb{Z}$ .

2. A water pitcher can be full, or empty, or something in between. If we let $w$ represent the amount of water that could be put in a one-quart pitcher, what numbers could $w$ 'hold'? Assume the numbers have units of 'quart' attached to them; that is, $\frac{1}{2}$ means '$\frac{1}{2}$ quart'.

3. Repeat (2), but this time assume the numbers have units of 'pint' attached to them. (There are two pints in one quart.)

Here's the precise definition of *variable*:

**DEFINITION**

*variable*

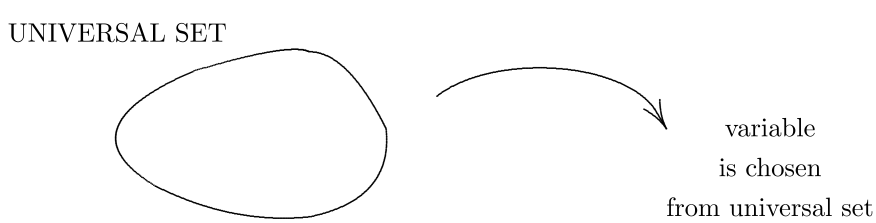
A *variable* is a symbol (usually a letter) that is used to represent a member of a specified set.

*universal set*

This 'specified set' is called the variable's *universal set*.

*Why the name 'variable'? Why the name 'universal set'?*

The *universal set* for a variable is its *supplier*—its *universe*: the variable is allowed to 'hold' anything that lives in its universal set. In other words, what the variable 'holds' is allowed to *vary* over the entire universal set; hence the name *variable* is appropriate.



*What is the universal set for a particular variable?*

Sometimes, a mathematician will be very forthright, and tell you exactly what a variable's universal set is. Other times, you have to decide what the universal set is yourself. This idea will be explored throughout the book.

*some common uses for variables*

Some common uses for variables are listed next, and investigated in examples 1–3:

- to state a general principle (Example 1)
- to represent a sequence of operations (Example 2)
- to represent something that is currently 'unknown', but that we would like to know (Example 3)

**EXAMPLE 1**

*variables are used to state a general principle*

Often, people need to state a property that is true for *so many individual cases* that it is impossible (or inconvenient) to list them all. Take, for example, a familiar property of addition: if you change the order in which two numbers are added, it doesn't affect the result. All the following are particular instances of this fact:

$$1+2=2+1 \qquad 2+3=3+2 \qquad 1.4+5.6=5.6+1.4 \qquad \tfrac{1}{2}+\tfrac{1}{3}=\tfrac{1}{3}+\tfrac{1}{2}$$

$$2+2=2+2 \qquad 0+\tfrac{1}{3}=\tfrac{1}{3}+0 \qquad 3+(-2)=(-2)+3 \qquad \ldots$$

Clearly, it's impossible to list *all* cases for which this is true. Here's how we can cover *all possible cases in one fell swoop:*

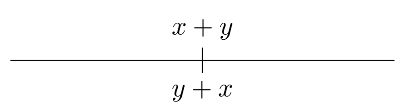
$$\text{For all real numbers } x \text{ and } y\,,\ \ x+y=y+x\,. \tag{*}$$

*understanding (*)*

Two variables appear in this true mathematical sentence: $x$ and $y$. The phrase

$$\text{For all real numbers } x \text{ and } y \ldots$$

informs us that the universal set for each variable is $\mathbb{R}$: so, $x$ can 'hold' any real number, and $y$ can also 'hold' any real number. But, no matter what numbers are being 'held' by $x$ and $y$, the sentence '$x+y=y+x$' is TRUE; that is, $x+y$ and $y+x$ are just different names for the same number. Notice that the use of variables has allowed us to say an *infinite number* of things, all at once.




★ *'For all' sentences*

Here's the full truth about 'For all' sentences:

Let $S(x)$ denote a statement about $x$, and consider the sentence

$$\text{For all } x\in U,\ \ S(x)\ . \tag{**}$$

If $S(x)$ is true for each and every $x\in U$, then (**) is true.

If there exists an $x\in U$ for which $S(x)$ is false, then (**) is false.

*more examples: using variables to state a general principle*

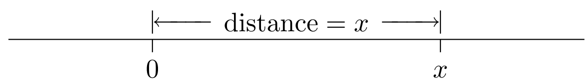
Here are more examples of using variables to state general principles, and some possible translations of these principles:

- For all real numbers $x$, $y$, and $z$, $(x + y) + z = x + (y + z)$.

Possible translation: When three numbers are being added, the grouping doesn't affect the result. You can group the first two numbers, then add the third; or, you can add the first number to the sum of the second and third. (See how awkward it is to try and say this in English!)

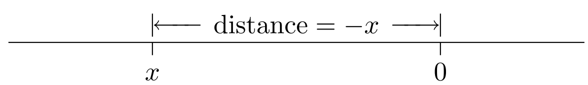
- For all nonnegative real numbers $x$, the distance between $x$ and zero on a number line is given by $x$.

Possible translation: If a number is zero, or lies to the right of zero on a number line, then its distance from zero is given by the number itself.




- For all negative real numbers $x$, the distance between $x$ and zero on a number line is given by $-x$.

Possible translation: If a number lies to the left of zero, then its distance from zero is given by its opposite. For example, the number $-3$ is 3 units from zero.




*read $-x$ as 'the opposite of $x$'*

Remember: if a number (like $x$) is negative, then its opposite (denoted by $-x$) is positive. It can be confusing to read the symbol $-x$ as 'negative ex', when 'negative ex' represents a positive number! For this reason, you might want to read $-x$ as 'the opposite of $x$' whenever confusion could otherwise result.

**EXERCISES**

4. Give a translation of this true mathematical sentence: For all real numbers $x$ and $y$, $x \cdot y = y \cdot x$.

5. How would a mathematician state the general principle that is being illustrated in the following cases?

$2 \cdot (3 \cdot 4) = (2 \cdot 3) \cdot 4 \qquad 7 \cdot (6 \cdot \frac{1}{2}) = (7 \cdot 6) \cdot \frac{1}{2} \qquad \frac{1}{2} \cdot (\frac{1}{3} \cdot \frac{1}{4}) = (\frac{1}{2} \cdot \frac{1}{3}) \cdot \frac{1}{4}$

$0 \cdot (1.2 \cdot \frac{1}{3}) = (0 \cdot 1.2) \cdot \frac{1}{3} \qquad -1 \cdot (3 \cdot 4) = (-1 \cdot 3) \cdot 4 \qquad \ldots$

*reading variables aloud*

When instructions are being given on how to read a mathematical sentence aloud, the following 'words' may be used to represent letters in the alphabet:

| letter in alphabet | 'word' used to represent the letter |
|---|---|
| $r$ or $R$ | 'arr' |
| $s$ or $S$ | 'ess' |
| $t$ or $T$ | 'tee' |
| $x$ or $X$ | 'ex' |
| $y$ or $Y$ | 'wye' |
| $z$ or $Z$ | 'zee' |

*sentences can be written in different ways*

Earlier, we encountered the sentence:

$$\text{For all real numbers } x \text{ and } y\,,\quad x+y=y+x\,.$$

The sentence could also be written like this:

$$\text{For all } x\in\mathbb{R} \text{ and } y\in\mathbb{R}\,,\quad x+y=y+x\,. \qquad (***)$$

Mathematics (like English) is flexible enough to allow for different *styles*: people can say the same thing in different ways. Most people would read the 'variation' (***) in one of the following ways:

- 'For all real numbers $x$ and real numbers $y$, $x$ plus $y$ equals $y$ plus $x$.' (Here, the sentence is being read with an understanding of what the symbols mean. If the reader's audience can't see the sentence, then this is the best way to read it.)
- 'For all ex in arr and wye in arr, $x$ plus $y$ equals $y$ plus $x$.' (If the reader's audience is looking at the sentence while it's being read, then this is the easiest way to read it.)

**EXERCISE**

6. As discussed above, give a variation for each of these sentences:
(a) For all real numbers $x$ and $y$, $x\cdot y=y\cdot x$.
(b) For all real numbers $x$, $y$, and $z$, $x\cdot(y\cdot z)=(x\cdot y)\cdot z$.

*context will determine the correct way to read '$x\in\mathbb{R}$'*

Look back at sentence (***). It would be extremely awkward (and incorrect) to read this sentence as:

'For all $x$ is in $\mathbb{R}$ and $y$ is in $\mathbb{R}$ ...'

Adjustments need to be made to the way that '$x\in\mathbb{R}$' is read, depending on its context. For example, in a 'for all ...' sentence, the word 'is' is dropped. Here's another example:

- Sentence: 'Let $x\in\mathbb{R}$.'

  How to read: 'Let ex be an element of arr' or 'Let ex be an element of the real numbers' or 'Let ex be a real number' or, most simply, 'Let ex be in arr'.

You choose your favorite way to read the sentence. Notice that in a 'let ...' sentence, the word 'is' is dropped, and the word 'be' is inserted.

Here's a summary of how someone could most concisely read '$x\in\mathbb{R}$' in different contexts:

| (self-standing) | | |
|---|---|---|
| $x\in\mathbb{R}$ | For all $x\in\mathbb{R}$ ... | Let $x\in\mathbb{R}$. |
| ex is in arr | For all ex in arr ... | Let ex be in arr. |

*different names for variables ALLOW for different choices, but don't REQUIRE different choices*

One final point before leaving the 'general principle' idea behind. Consider again the sentence:

$$\text{For all real numbers } x \text{ and } y\,,\quad x+y=y+x\,.$$

Just because the letter $x$ is different from the letter $y$ doesn't mean they *have* to 'hold' different numbers. Both $x$ and $y$ could equal $3$, for instance, yielding the true sentence $3+3=3+3$. Using different letters *allows* for different choices, but does not *require* different choices.

*a shorthand for denoting multiplication*

*$xy$ means $x \cdot y$*

*$2x$ means $2 \cdot x$*

Before proceeding to Example 2, it's necessary to discuss a shorthand used in mathematics to denote multiplication, when variables are involved:

| | | |
|---|---|---|
| $x \cdot y$ | gets shortened to | $xy$ |
| $2 \cdot x$ | gets shortened to | $2x$ |
| $x \cdot y \cdot z$ | gets shortened to | $xyz$ |

That is, when no confusion can result, the centered dot that denotes multiplication is usually dropped. Of course, $2 \cdot 3$ *cannot* be shortened to $23$, since this would be confused with the number twenty-three!

*convention:*

*write $2x$, not $x2$*

Furthermore, whenever a variable (like $x$) is being multiplied by a specific number (like $2$), it is conventional to write the specific number *first*. That is, write $2x$, NOT $x2$. As you read mathematics, start noticing that whenever a specific number is multiplying a variable, the specific number is always written first.

**EXERCISE**

7. Give shorthands, if possible, for each of the following expressions. Write each in the most conventional way.

$$a \cdot b \qquad x \cdot 3 \qquad a \cdot 5 \cdot c \qquad 3 \cdot 4$$

**EXAMPLE 2**

*variables are used to represent a sequence of operations*

One of the most common uses of variables is to represent a sequence of operations. Consider the following request:

'Take a number, multiply by $2$, then add $1$.'

$$x \xrightarrow{\text{multiply by } 2} 2x \xrightarrow{\text{add } 1} 2x+1$$

*mapping diagram*

The diagram above is sometimes called a 'mapping diagram'; it can be used to represent a sequence of operations.

If we let the original number be denoted by $x$, then:

- multiplying $x$ by 2 gives the result $2x$; then,
- adding 1 gives the result $2x + 1$.

Thus, the expression $2x + 1$ represents the sequence of operations: 'take a number, multiply by $2$, then add $1$'.

*changing the order in which the operations are applied*

Suppose the previous example is modified slightly, by changing the order in which the operations are applied. This time, we want to represent the sequence: 'Take a number, add $1$, then multiply by $2$.'

$$x \xrightarrow{\text{add } 1} x+1 \xrightarrow{\text{multiply by } 2} 2(x+1)$$

Again denoting the original number by $x$:

- adding 1 to $x$ gives $x + 1$; then,
- multiplying by 2 gives $2(x + 1)$.

The parentheses are required, because we want to take the entire quantity $x+1$ and multiply it by $2$. The expression $2(x + 1)$ represents the sequence of operations: 'take a number, add $1$, then multiply by $2$'.

*table comparing the expressions $2x + 1$ and $2(x + 1)$*

The table below compares the expressions $2x+1$ and $2(x+1)$ for several whole numbers $x$. Observe that the results are different!

| $x$ (the input) | $2x + 1$ (multiply by $2$, then add $1$) | $2(x + 1)$ (add $1$, then multiply by $2$) |
|---|---|---|
| 0 | $2 \cdot 0 + 1 = 1$ | $2(0 + 1) = 2$ |
| 1 | $2 \cdot 1 + 1 = 3$ | $2(1 + 1) = 4$ |
| 2 | $2 \cdot 3 + 1 = 7$ | $2(3 + 1) = 8$ |

*reading the expressions $2x + 1$ and $2(x + 1)$ aloud*

How should each of the expressions $2x+1$ and $2(x+1)$ be read aloud? Again, we'll see ourselves running into an aforementioned problem: mathematics is primarily designed to be a written language, not a spoken language. If the audience is *looking* at the written expressions while they're being read aloud, then this will work:

| expression | how to read aloud |
|---|---|
| $2x + 1$ | two ex plus one |
| $2(x + 1)$ | two times (*slight pause*) ex plus one |

The pause is supposed to clue the listener that the 'ex plus one' is grouped together. Unfortunately, though, it's difficult to 'hear' a pause. So, if the audience *is not* looking at the expression while it's being read, then it's better to say:

| expression | how to read aloud |
|---|---|
| $2(x + 1)$ | two times the quantity ex plus one<br>or<br>two times, open parenthesis, ex plus one, close parenthesis |

In the first case, the words 'the quantity' clue the listener that the 'ex plus one' is grouped together. The second case is a 'verbatim' reading of the symbols.

*What is done to x FIRST?*

When you look at an expression (like $2x + 1$) with the goal of 'translating' it into a sequence of operations, you must start by asking: 'What is done to $x$ *first*?' Sometimes this question is easy to answer; other times not.

*brief introduction to the 'order of operations' conventions*

The *order of operations conventions* are needed for a complete answer to the question 'What's done first?' Here's a quick, informal, introduction to these conventions.

When two operations (say, addition and multiplication) are 'competing' for the same number, a decision needs to be made about 'who will win'. To illustrate this idea, consider the expression $1 + 2x$, which has been typeset below in an unusual way:

$$1 \;\overset{\leftarrow\;\rightarrow}{+}\; 2 \;\overset{\leftarrow\;\rightarrow}{\cdot}\; x$$

Think of the arrows above each operation as 'arms' that are trying to grab the numbers the operation needs to work with. The addition operation is trying to 'grab' the numbers 1 and 2. The multiplication operation is trying to 'grab' the numbers 2 and $x$. Notice that *both operations* are trying to 'grab' the number 2. Who will win?

I'll tell you the answer: multiplication wins. It has been decided that *multipication is 'stronger than' addition.* This of course makes sense, since multiplication can be viewed as 'super-addition': after all, $5 \cdot 2$ means five piles of two $(2 + 2 + 2 + 2 + 2)$ or two piles of five $(5 + 5)$. (That's all you're going to get about the order of operations conventions in this book! Look in an algebra book for a more complete discussion.)

*going from an expression to a sequence of operations; and vice versa*

Whenever you see a simple expression (like $2x + 1$), a *sequence of operations* needs to jump into your mind! Also, whenever you need to represent a sequence of operations, you must be able to write a mathematical expression representing the sequence. The next exercise gives you practice with both directions.

**EXERCISES**

8. Represent each of the following sequences of operations by an expression. Let $x$ denote the number that you're starting with. Use a horizontal fraction bar to denote division: that is, use the symbol $\frac{x}{y}$ to denote $x$ divided by $y$.

(a) Take a number, multiply by $3$, then subtract $4$.

(b) Take a number, subtract $4$, then multiply by $3$.

(c) Take a number, divide by $2$, then add $1$.

(d) Take a number, add $1$, then divide by $2$.

9. In words, describe the sequence of operations represented by each expression:

a) $5x - 3$ (c) $\frac{x}{4} - 1$

b) $5(x - 3)$ (d) $\frac{x-1}{4}$

**EXAMPLE 3**

*variables can represent an 'unknown' quantity*

Very often in life, you know something *about* a number, without knowing (at least initially) exactly what the number *is*. Here are a couple examples:

*clothes sale*

There's a sale at your favorite clothes store. First, everything was discounted by $30\%$. Now, they're taking an additional 20% off the previous sale prices. You've got \$100 in your clothes budget. Since you're anticipating a *big crowd*, you want to be prepared to grab the clothes and head for the register. Accounting for 5% sales tax, in addition to the discounts, how many dollars worth of clothes can you bring to the register? (You'll be totaling up the *original* prices on the tags; before any discounts.)

*passing cars*

The author of this book usually drives the speed limit. Consequently, she often finds herself being passed by other cars on the freeway. Just how fast *are* those other cars going when they whiz by? The author knows the length of her own car. She can estimate the number of seconds it takes for the front of the passing car to travel from her rear bumper to her front bumper. Can these numbers be put together to estimate how fast the other car is going?

*general approach for 'something is unknown' problems*

Here's a general approach for solving problems like those mentioned above:

- A *name* is assigned to the thing you want to know (but don't initially know). This is your *variable*! Try to choose a name that suggests what it represents: $p$ is a good choice for representing a *price*; $s$ is good for representing a *speed*; $m$ is good for representing the number of *minutes* you may talk on a telephone each month. Often, $x$ is used if no other choice seems obvious.

- Write a *mathematical sentence* involving the variable, which is *true* when the variable takes on the desired value(s). Initially, the sentence may look very complicated, and it may not be the least bit obvious what value(s) of the variable make the sentence true.

- Find the choice(s) for the variable that make your sentence *true*. Some techniques for doing this will be discussed in the last few sections of this book.

**EXERCISES**

10. What might be a good variable to represent each of the following 'unknowns'?
    (a) an unknown distance
    (b) an unknown time
    (c) an unknown shoe size
    (d) an unknown volume

*illustrating the procedure step-by-step*

The procedure just described is illustrated, step-by-step, for the 'clothes sale' problem. The 'passing cars' problem is answered in a later section.

Depending on your math background, some parts of the following discussion may seem a bit vague. Remember: the goal here is to focus on the general problem-solving approach, and the role that a variable plays in this approach.

*several 'percent' ideas*

There are several 'percent' ideas needed to formulate the clothes sale problem:

- If you buy an item that is discounted 30% , then you're left paying 70% of the original price. (Similarly, 20% off leaves you paying 80% .)
- If $x$ is any number, then the new number ' 70% of $x$ ' goes by the name ' $0.7x$ '.
- Five percent sales tax gets handled like this:

$$\overbrace{x}^{\text{selling price}} + \overbrace{0.05x}^{\text{tax is 5\% of } x} = \overbrace{1.05x}^{\text{price, including tax}}$$

Now, we're ready to formulate the clothes sale problem.

*STEP 1: give a name to the unknown*

STEP 1 (give a name to the unknown): Let $p$ (for 'price') represent the total non-discounted dollar amount of clothes you bring to the register. (That is, the clerk adds up all the original, non-discounted, prices of your items, giving the result $p$ .)

*STEP 2: write a mathematical sentence involving the unknown*

STEP 2 (write a mathematical sentence involving $p$): Here's what the sales clerk does with the total of the original prices ($p$):

- The 30% discount leaves you owing 70% of the original amount: $0.7p$
- The additional 20% discount leaves you owing 80% of the prior amount: $(0.8)(0.7p)$
- The 5% sales tax on the prior amount leaves you owing $(1.05)(0.8)(0.7p)$ .
- You have \$100 to spend (and you're willing to spend it all!) In other words, you want the amount you owe ( $(1.05)(0.8)(0.7p)$ ) to equal the amount you have (100) :

$$\overbrace{(1.05)(0.8)(0.7p)}^{\text{amount you owe}} \quad = \quad \overbrace{100}^{\text{amount you have}}$$

Replace the name $(1.05)(0.8)(0.7p)$ with the simpler name $0.588p$ . (Check this with a calculator. Read '$0.588p$' as 'point five eight eight pee'.)

*analysis of the sentence '$0.588p = 100$'*

Let's analyze the sentence '$0.588p = 100$'. This sentence is *false* for *lots* of values of $p$ . Use your calculator to check the results below:

| $p$ | substitution into sentence '$0.588p = 100$' | resulting sentence is ... |
|---|---|---|
| 100 | $0.588(100) = 100$ <br> $58.8 = 100$ | false |
| 150 | $0.588(150) = 100$ <br> $88.2 = 100$ | false |
| 160 | $0.588(160) = 100$ <br> $94.08 = 100$ | false |
| 180 | $0.588(180) = 100$ <br> $105.84 = 100$ | false |

What we want to know is: What number $p$ makes this sentence *true*?

*STEP 3: figure out when the sentence is TRUE*

STEP 3 (figure out when the sentence is true): The sentence '$0.588p = 100$' is not particularly convenient to work with, from the point of view of determining when it is *true*. In this form, you must think: What number, when multiplied by 0.588 , gives 100 ? So, we'll 'transform' the sentence into one that's easier to work with. There are certain things that you can *do* to sentences, that will make them *look* different, but that won't change when they're true, or when they're false. These 'transforming tools' are the topic of later sections—just a brief preview is given here.

Think about this: if two numbers are equal, and you divide them both by the same thing, will the resulting numbers still be equal? Sure!

And, if two numbers are *not* equal, and you divide them both by the same thing, will they remain nonequal? Sure!

So, if we divide both sides of the sentence '$0.588p = 100$' by the same number, then we'll end up with a sentence that *looks* different, but that has the same truth values as the sentence we started with!

*the transforming process*

Here's the 'transformation':

$$0.588p = 100 \qquad \text{(original sentence)}$$

$$\frac{0.588p}{0.588} = \frac{100}{0.588} \qquad \text{(divide both sides by 0.588)}$$

$$p = 170.0680272\ldots \qquad \text{(much easier sentence to work with!)}$$

This final sentence is so simple that it *tells* you when it's true!
(Check, with your calculator, that $\frac{100}{0.588} = 170.0680272\ldots$ .)

So—you could bring about \$170 worth of clothes to the counter! Any more, and you won't have enough money. Any less, and you'll get some change.

**EXERCISE**

11. Suppose the original prices on your clothes items total \$170 .

(a) After the 30% discount, how much do you owe?
(Using a calculator, compute $(0.7)(170)$ . )

(b) After the additional 20% discount, how much do you owe?
(Using a calculator, compute $(0.8)$(previous answer). )

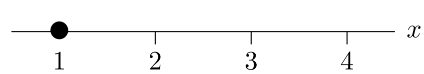
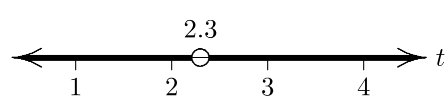
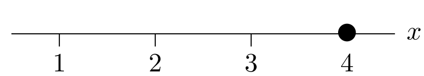
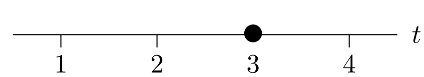
(c) After 5% sales tax, how much do you owe?
(Using a calculator, compute $(1.05)$(previous answer). )

(d) How much change will you get from \$100 ?

*solving a sentence;*

*solving a sentence by inspection*

The process of determining when a sentence is *true* is called '*solving the sentence*'. Some sentences are so simple that they can be solved *by inspection*; that is, you can just look at them and decide when they're true:

| sentence | choice(s) that make the sentence true |
|---|---|
| $x = 1$ |  |
| $t \ne 2.3$ |  |
| $x - 1 = 3$ |  |
| $2t = 6$ |  |

*English sentences to guide your thought process*

Each of these mathematical sentences has a corresponding English sentence that guides the thought process used to determine when the sentence is true. These English sentences are discussed next:

| mathematical sentence | corresponding English sentence |
|---|---|
| $x = 1$ | What number equals 1? ANSWER: Only the number 1 . |
| $t \neq 2.3$ | What numbers do *not* equal 2.3 ? ANSWER: all real numbers except 2.3 . |
| $x - 1 = 3$ | What number, minus 1 , gives 3 ? ANSWER: the number 4 . |
| $2t = 6$ | What number, times 2 , gives 6 ? ANSWER: the number 3 . |

Observe that there are varying degrees of 'simple'. In the examples above, the first two sentences are somewhat 'simpler' than the last two.

**EXERCISE**

12. For each sentence below, write an English sentence reflecting the thought process you use to determine when the sentence is true.

Then, solve each sentence 'by inspection'. That is, look at the sentence, stop and think, and determine when the sentence is true.

(a) $t = 5$
(b) $x \neq 2$
(c) $3x = 12$
(d) $\frac{x}{3} = 4$
(e) $x + x + x = 12$
(f) $2 + t = 2 - t$
(g) $\frac{15}{x} = 3$
(h) $12 - y - y = 10$

*conventions*

As mentioned in the first section of this book, English has lots of conventions. For example, the capitalization of proper nouns clues the reader that 'Carol' refers to a person, whereas 'carol' refers to a Christmas song.

Mathematics has lots of conventions regarding the naming of variables, which help clue the reader to the type of objects the variable can 'hold'.

*numbers: lowercase letters*

*Numbers* are usually represented by lowercase letters (like $a$ , $n$ , or $x$ ). Go back to the beginning of this book, and leaf through the pages, looking for places where a letter has been used to represent a number. Lowercase letters were used!

*sets: uppercase letters*

*Sets* are usually represented by uppercase letters (like $A$ , $B$ , or $S$ ). Notice that the symbol $\mathbb{R}$ (used to represent the set of real numbers) and the symbol $\mathbb{Z}$ (used to represent the set of integers) are both uppercase letters (in a special typestyle).

So—numbers are usually represented by lowercase letters, and sets by uppercase letters. The conventions go even further than this ...

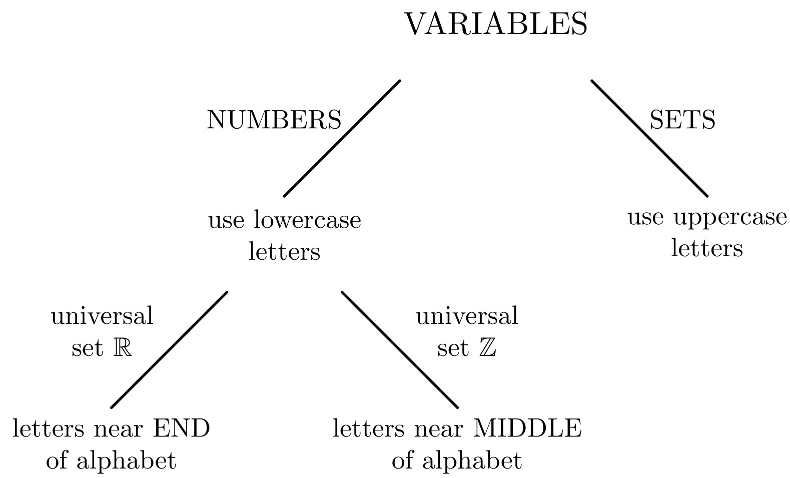
*universal set $\mathbb{R}$: letters near end of alphabet*

A variable with universal set $\mathbb{R}$ (or, any *interval* of real numbers) is most likely to be named with a lowercase letter from the *end* of the alphabet, particularly $t$ , $x$ , or $y$ .

*universal set $\mathbb{Z}$: letters near middle of alphabet*

A variable with universal set $\mathbb{Z}$ (or, any subset of $\mathbb{Z}$) is most likely to be named with a lowercase letter near the *middle* of the alphabet; particularly $i$ , $j$ , $k$ , $m$ , or $n$ . (MEMORY DEVICE: The <u>in</u>tegers use letters from $i$ to $n$ .)

These conventions are summarized in the diagram below:




**EXERCISES**

13. For each pair of mathematical sentences given below, choose the 'best' one, in keeping with normal mathematical conventions. Also state how you might read the 'best' choice aloud.

| | | |
|---|---|---|
| (a) | Let $x \in \mathbb{R}$ . | Let $k \in \mathbb{R}$ . |
| (b) | Let $x \in \mathbb{Z}$ . | Let $k \in \mathbb{Z}$ . |
| (c) | For all $t \in [0,2]$ ... | For all $i \in [0,2]$ ... |
| (d) | For all $t \in \{1,2,3,\dots\}$ ... | For all $i \in \{1,2,3,\dots\}$ ... |

14. For each variable below, give as much information as possible about what the variable probably 'holds'. Use normal mathematical conventions to reach your conclusions.

(a) $b$ (b) $B$ (c) $t$
(d) $k$ (e) $S$ (f) $x$

*overriding conventions*

Remember that the ultimate choice of letter used for a variable is *up to you.* If you really want to call your 'unknown' price $P$ (uppercase) instead of $p$ (lowercase), go ahead! You can override any convention simply by informing people of your intentions.

Keep in mind, however, that an awareness of mathematical conventions provides valuable clues in reading mathematics. And, if you (mostly) adhere to normal conventions in your own mathematical work, then you're apt to have a happier audience.

*variables are typeset in an italic typestyle*

Whenever letters are used in a mathematical context (i.e., as variables), they are typeset in an italic style. This convention helps to visually distinguish letters being used in a mathematical way from letters being used in a non-mathematical way. Again go back to the beginning of this book, and look carefully at the way that variables appear: you'll see that an italic typestyle is used.

*hand-writing variables*

When hand-writing mathematics, it's particularly easy to confuse variables with other things, as the following cautions indicate:

DON'T write $x$ as  ; it can be confused with a multiplication symbol.

DON'T write $y$ as  ; it can look like an ex.

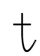
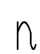
DON'T write $z$ as  ; it can look like the number two.

DON'T write $t$ as  ; it can look like a plus sign.

DON'T write $i$ as  ; dots get lost, and then it looks like the number one.

DON'T write $l$ as  ; it can look like the number one.

For these reasons, when hand-writing mathematics, you'll want to try and 'duplicate' an italic font, as illustrated in the following table:

| typestyle used in English words | typestyle used for variable | how to hand-write |
|---|---|---|
| 'x' (as in 'except') | $x$ (as in $x + y$) |  |
| 'y' (as in 'yes') | $y$ (as in $y + x$) |  |
| 'z' (as in 'zoo') | $z$ (as in $z + x$) |  |
| 't' (as in 'top') | $t$ (as in $t + x$) |  |
| 'i' (as in 'it') | $i$ (as in $i + 1$) |  |
| 'j' (as in 'jog') | $j$ (as in $j + 1$) |  |
| 'k' (as in 'kit') | $k$ (as in $k + 1$) |  |
| 'l' (as in 'let') | $l$ (as in $l - 1$) |  |
| 'm' (as in 'men') | $m$ (as in $m - 1$) |  |
| 'n' (as in 'no') | $n$ (as in $n - 1$) |  |

**EXERCISE**

15. Trace the following, to practice writing variables in the correct way:

x x x x x i i i i i l l l l l

y y y y y j j j j j m m m m m

z z z z z k k k k k n n n n n

t t t t t

**END-OF-SECTION EXERCISES**

For exercises 16–26: Classify each entry as a mathematical expression (EXP), or a mathematical sentence (SEN).

If an expression, state whether it is a number or a set.

Classify the truth value of each sentence: (always) true (T); (always) false (F); or sometimes true/sometimes false (ST/SF).

16. $xyz$ ________________
17. $xyz = zyx$ ________________
18. $5x + 1$ ________________
19. $5x + 1 = 1 + 5x$ ________________
20. $[0, 5)$ ________________
21. $5 \in [0, 5)$ ________________
22. $\frac{x}{4} - 1$ ________________
23. $5(0.15) = 0.75$ ________________
24. $\{0, 5\}$ ________________
25. $5 \in \{0, 5\}$ ________________
26. $(1.05)(0.8)(0.7p) = 100$ ________________

For each mathematical sentence below, write an English sentence that shows the thought process behind determining when the sentence is true. Then, solve the sentence by inspection.

If there is more than one number that makes the sentence true, then shade the numbers that make the sentence true on a number line.

The first one is done for you.

(sample): $\frac{x}{4} = 5$ : What number, when divided by $4$, gives $5$? Answer: 20

27. $\frac{20}{x} = 5$

28. $20 - x = 2$

29. $3x = 4x$

30. $3x \neq 4x$

31. $x + 1 \neq x + 2$

# SECTION SUMMARY

## HOLDING THIS, HOLDING THAT

| NEW IN THIS SECTION | HOW TO READ | MEANING |
|---|---|---|
| variable<br>universal set | | A variable is a symbol (usually a letter) used to represent a member of a specified set. This specified set is called the variable's *universal set*. |
| common uses for variables | | to state a general principle;<br>to represent a sequence of operations;<br>to represent an 'unknown' |
| reading letters aloud | 'arr' represents $r$ or $R$<br>'ess' represents $s$ or $S$<br>'tee' represents $t$ or $T$<br>'ex' represents $x$ or $X$<br>'wye' represents $y$ or $Y$<br>'zee' represents $z$ or $Z$ | 'words' used to represent letters in the alphabet, when discussing how to read a mathematical sentence |
| For all real numbers $x$ and $y$ ...<br>For all $x \in \mathbb{R}$ and $y \in \mathbb{R}$ ... | | different ways to say the same thing |
| $x \in \mathbb{R}$<br>For all $x \in \mathbb{R}$<br>Let $x \in \mathbb{R}$ | 'ex is in arr'<br>'For all ex in arr'<br>'Let ex be in arr' | Context will determine the correct way to read '$x \in \mathbb{R}$'. |
| $xy$ | 'ex wye' (preferred)<br>or<br>'$x$ times $y$' | a shorthand for $x \cdot y$; when no confusion can result, the centered dot that denotes multiplication can be dropped |
| $2x$ | 'two ex' (preferred)<br>or<br>'two times ex' | whenever a variable is being multiplied by a specific number, write the specific number *first* |
| mapping diagram<br>$x \xrightarrow{\text{multiply by } 2} 2x \xrightarrow{\text{add } 1} 2x+1$ | | a diagram that can be used to represent a sequence of operations |
| $2x+1$ | 'two ex plus one' | denotes the sequence of operations: take a number, multiply by $2$, then add $1$ |
| $2(x+1)$ | 'two times<br>the quantity ex plus one' | denotes the sequence of operations: take a number, add $1$, then multiply by $2$ |
| solving a sentence | | the process of determining when a sentence is *true* |

# SECTION SUMMARY

## HOLDING THIS, HOLDING THAT

| NEW IN THIS SECTION | HOW TO READ | MEANING |
|---|---|---|
| solving a sentence by inspection | | Looking at a sentence, stopping and thinking, and determining when the sentence is true. |
| lowercase letters (like $a$, $n$, $x$) | | *numbers* are usually represented by lowercase letters |
| lowercase letters from end of alphabet (particularly $t$, $x$, $y$) | | a variable with universal set $\mathbb{R}$ (or, any *interval* of real numbers) is most likely to be named with a lowercase letter from the *end* of the alphabet |
| lowercase letters from middle of alphabet (particularly $i$, $j$, $k$, $m$, $n$) | | a variable with universal set $\mathbb{Z}$ (or, any subset of $\mathbb{Z}$) is most likely to be named with a lowercase letter from the *middle* of the alphabet |
| uppercase letters (like $A$, $B$, $S$) | | *sets* are usually represented by uppercase letters |
| hand-writing variables<br>j k l m n | write x , NOT x<br>write y , NOT x<br>write z , NOT z<br>write t , NOT t<br>write i , NOT i<br>write l , NOT l | Try to duplicate an italic typestyle when hand-writing variables, to prevent confusion. |