7. THESE SENTENCES CERTAINLY LOOK DIFFERENT

comparing expressions versus comparing sentences

Whereas the '=' sign gives a way to compare mathematical *expressions* (like numbers and sets), the idea of *equivalence* gives a way to compare mathematical *sentences*.

a motivating example: sentences that LOOK different; but, in a very important way, are 'the same'

To motivate the idea of equivalence, consider these two mathematical sentences:

$$\text{`} 2x - 3 = 0 \text{'} \qquad \text{and} \qquad \text{`} x = \tfrac{3}{2} \text{'}$$

They certainly *look* different. The first sentence has a zero on one side; the second doesn't. The second sentence involves a fraction; the first doesn't.

But, in a very important way, these two sentences are 'the same': no matter what real number is chosen for the variable x, these two sentences *always have the same truth values*.

For example, choose x to be $\frac{3}{2}$.

Substitution into '$2x - 3 = 0$' yields the TRUE sentence '$2(\frac{3}{2}) - 3 = 0$'.

Substitution into '$x = \frac{3}{2}$' yields the TRUE sentence '$\frac{3}{2} = \frac{3}{2}$'.

Next, choose x to be 5.

Substitution into '$2x - 3 = 0$' yields the FALSE sentence '$2(5) - 3 = 0$'.

Substitution into '$x = \frac{3}{2}$' yields the FALSE sentence '$5 = \frac{3}{2}$'.

No matter what real number is chosen for x, these two sentences will ALWAYS have the same truth values.

The previous examples (and some more!) are summarized in the table below:

x	into '$2x - 3 = 0$'	first sentence is ...	into '$x = \frac{3}{2}$'	second sentence is ...
$\frac{3}{2}$	$2(\frac{3}{2}) - 3 = 0$	true	$\frac{3}{2} = \frac{3}{2}$	true
5	$2(5) - 3 = 0$	false	$5 = \frac{3}{2}$	false
0	$2(0) - 3 = 0$	false	$0 = \frac{3}{2}$	false
2	$2(2) - 3 = 0$	false	$2 = \frac{3}{2}$	false
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Indeed, '$2x - 3 = 0$' is true only when x is $\frac{3}{2}$, and false otherwise.

Also, '$x = \frac{3}{2}$' is true only when x is $\frac{3}{2}$, and false otherwise.

sentences that always have the same truth values can be used interchangeably

When two mathematical sentences always have the same truth values, then they can be used *interchangeably*, and you can use whichever sentence is easiest for a given situation.

Why might one 'form' of a sentence be easier to use than another? Well, the sentence '$x = \frac{3}{2}$' is easier than the sentence '$2x - 3 = 0$' in a very important way: it is easy to just *look* at '$x = \frac{3}{2}$' and *see* the value that makes it true. You only need to think: 'What number is equal to $\frac{3}{2}$?' That is, the sentence '$x = \frac{3}{2}$' can be solved *by inspection*.

On the other hand, to decide what number makes '$2x - 3 = 0$' true, you must think: 'What number, when doubled, then with 3 subtracted from the result, gives zero?' Much harder!

mathematical verb for comparing sentences: 'is equivalent to'	The mathematical verb used to compare the truth values of sentences is: 'is equivalent to'.
	Here's how you could be told that the sentences '$2x - 3 = 0$' and '$x = \frac{3}{2}$' always have the same truth values:
	For all real numbers x, $2x - 3 = 0$ is equivalent to $x = \frac{3}{2}$.
	You should read this sentence aloud as:
	For all real numbers ex, two ex minus three equals zero (slight pause) is equivalent to (slight pause) ex equals three-halves.
	Your translation of this sentence should be:
	No matter what real number is substituted for x, the sentences '$2x - 3 = 0$' and '$x = \frac{3}{2}$' will have the same truth values. For a given value of x, if one sentence is true, so is the other; if one sentence is false, so is the other.

★	Many books define two sentences with one free variable to be *equivalent* when they have the same solution sets: hence, such books would merely say that '$2x - 3 = 0$ is equivalent to $x = \frac{3}{2}$' (that is, they would leave off the 'for all' part of the sentence). It will become clear, after defining 'is equivalent to' via a truth table later in this section, why the author has included the 'for all' here. Furthermore, at the conclusion of this section, the author discusses situations under which the words 'for all' can be omitted.

	Even though the sentences '$2x - 3 = 0$' and '$x = \frac{3}{2}$' *look* different, they are completely interchangeable from a 'truth' point of view: they are true at the same time, and false at the same time.
	The purpose of this section is to make the idea of mathematical equivalence even more precise.
'equal' and 'equivalent' are NOT synonyms in mathematics!	First of all, it's important to discuss the difference between the words 'equal' and 'equivalent'. In English, these two words tend to be used somewhat interchangeably; but in mathematics, they have *very different uses*!
	In mathematics, the words 'equal' and 'equivalent' are NOT synonyms!
	Equality refers to the state of being equal; **equivalence** refers to the state of being equivalent.
	In mathematics, 'equality' and 'equivalence' are NOT synonyms!
When do we use 'equal'? When do we use 'equivalent'?	How are you going to know when it is appropriate to use 'equal', and when it is appropriate to use 'equivalent'? The key lies in what is being *compared*. Generally,
	You compare EXPRESSIONS using 'EQUAL'.
	You compare SENTENCES using 'EQUIVALENT'.
MEMORY DEVICE	Notice that if you take the first letter of 'Sentence' (S) and put it together with the last three letters of 'equivalENT' (ENT), then you get 'SENT' (as in SENTence). Perhaps this will be enough to remind you that mathematicians generally talk about
	$\underline{S}$entences being equival$\underline{ENT}$.
What does it mean for expressions to be equal?	What does it mean for expressions to be equal? It depends on the type of expression:
	• When two *numbers* are *equal*, then they live at precisely the same place on a number line.
	• When two *sets* are *equal*, then they have precisely the same members.

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*What does it mean
for sentences
to be equivalent?*

What does it mean for sentences to be equivalent? This has to do with the *truth values* of the sentences being the same!

*It DOESN'T MAKE
SENSE to talk about
sentences being 'equal'!*

Why doesn't it make sense to talk about *sentences* being 'equal'?

Well, consider this pair of sentences:

$$`\,x = 4\,` \qquad \text{and} \qquad `\,x - 4 = 0\,`$$

Each is true when x is 4, and false otherwise. Suppose we were to say (as some students inadvertently do) that these sentences are 'equal'. Then, we'd be faced with this absurdity:

$$\underbrace{x = 4}_{\text{first sentence}} \quad \overbrace{=}^{\text{is equal to?}} \quad \underbrace{x - 4 = 0}_{\text{second sentence}}$$
$$x = 4 = x - 4 = 0 \;\; ???$$

The result: we're being told that x is equal to 4, which is equal to $x - 4$, which is equal to 0. In particular, we're being told that 4 is equal to 0. How absurd! How *false*!

In mathematics, we generally talk about:

<div align="center">

EXPRESSIONS being EQUAL

SENTENCES being EQUIVALENT

</div>

EXERCISES

1. Look up the word 'equal' in an ENGLISH dictionary. Is the word 'equivalent' used in the definition?

2. Look up the word 'equivalent' in an ENGLISH dictionary. Is the word 'equal' used in the definition?

3. Would your answers to questions 1 and 2 lead you to believe that, in English, 'equal' and 'equivalent' tend to be viewed as synonyms?

4. In mathematics, are 'equal' and 'equivalent' synonyms? Comment.

★
*Can expressions be
'equivalent'?*

Most often, mathematicians talk about *expressions* being *equal*. (*Numbers* can be equal; *sets* can be equal; *functions* can be equal; *ordered pairs* can be equal; *matrices* can be equal; *vectors* can be equal; etc.) However, there are a few situations where mathematicians also talk about expressions being *equivalent*. Here's one.

Warren Esty, in his excellent book **The Language of Mathematics**, defines two variable numerical expressions to be *equivalent* if they are equal for all possible value(s) of the variable(s).

For example, with his definition, the expressions $2x$ and $x + x$ are equivalent, because they are *always* equal.
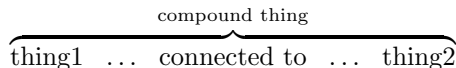
This author prefers to reserve the word 'equivalent' for comparing *sentences only*. So, how can we express the fact that $2x$ and $x + x$ are always equal? Like this:

$$\text{For all real numbers } x\,,\; 2x = x + x\,.$$

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connectives;
compound sentences

'Equivalence' has not yet been discussed with the needed precision. To this end, we must next investigate 'connectives' and 'compound sentences'.

Mathematicians frequently take 'little' things and connect them into 'bigger' things, using appropriate **connectives**. Once connected up, the result is often referred to as a **compound** thing:

$$\overbrace{\text{thing1} \ \dots \ \text{connected to} \ \dots \ \text{thing2}}^{\text{compound thing}}$$

different types
of connectives,
depending on what
is being connected

There are different types of connectives, depending on what is being connected. Expressions can be connected to get 'compound' expressions; sentences can be connected to get 'compound' sentences. Some examples follow:

EXAMPLE

connectives
for numbers

What can be done with two numbers, that gives another number? Here, the four most common connectives for numbers are discussed:

Let x and y be real numbers.

The operation '$+$' (addition) is a connective for numbers: x and y can be 'connected' to give $x + y$.

The operation '$-$' (subtraction) is a connective for numbers: x and y can be 'connected' to give $x - y$.

The operation '$\cdot$' (multiplication) is a connective for numbers: x and y can be 'connected' to give $x \cdot y$ (more simply written as xy).

The operation '$/$' (division) is a connective for numbers: as long as $y \neq 0$, then x and y can be 'connected' to give x/y (also written as $\frac{x}{y}$ or $x \div y$).

★

connectives for sets

Two of the most common connectives for sets are $\cup$ (union) and $\cap$ (intersection). Thus, if A and B are sets, then $A \cup B$ and $A \cap B$ are also sets.

EXAMPLE

the English word 'and'
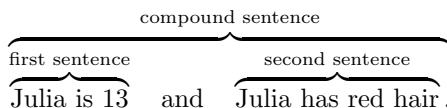*can be used to*
connect sentences

The word 'and' can be used as a connective for *sentences* in the English language, as discussed next.

Suppose that Julia is a 13-year-old red-haired girl. Then, both of these sentences are true:

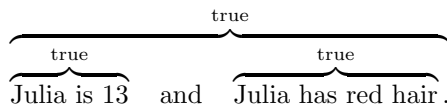'Julia is 13.'

'Julia has red hair.'

These two sentences can be combined into a 'compound' sentence using the English word 'and':

$$\overbrace{\overbrace{\text{Julia is 13}}^{\text{first sentence}} \quad \text{and} \quad \overbrace{\text{Julia has red hair}}^{\text{second sentence}}}^{\text{compound sentence}}.$$

(Notice that a 'compound sentence' is still a *sentence*: it expresses a complete thought; it makes sense to ask about its truth.)
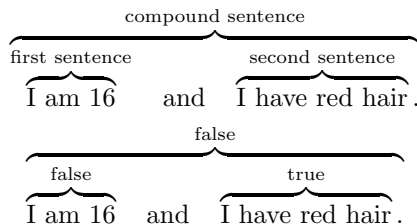
When is an 'and'
sentence TRUE?

Is the compound sentence true or false? In this case, the compound sentence is true. This is because, in English, an 'and' sentence is true when both of the subsentences are true:

$$\overbrace{\overbrace{\text{Julia is 13}}^{\text{true}} \quad \text{and} \quad \overbrace{\text{Julia has red hair}}^{\text{true}}}^{\text{true}}.$$

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When is an 'and'
sentence FALSE?

On the other hand, suppose Julia were to say (with a bit of a smirk):

She'd be lying! That is, the compound sentence is false. In English, an 'and' sentence is false when at least one of the subsentences is false. That is, an 'and' sentence is false when one, or the other, or both, of the subsentences are false.

in English,
'and' can be used
as a connective
for sentences;
as a connective
for nouns

It is interesting to note that the English word 'and' can also be used as a connective for nouns. For example: 'cat' is a noun; 'dog' is a noun; 'cat and dog' is a compound noun.

Thus, the English word 'and' is extremely versatile: it can be used as a connective for sentences, and it can also be used as a connective for nouns. (We'll soon see that the mathematical word 'and' is not quite so versatile!)
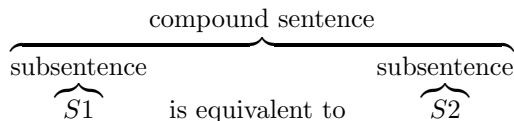

EXERCISE

5. Consider the sentence:

Thus, the English word 'and' is extremely versatile: it can be used as a connective for sentences, **AND** *it can also be used as a connective for nouns.*

How is the bold-faced 'and' being used in this sentence? As a connective for sentences, or as a connective for nouns?

---

the truth
of a compound sentence
depends on the truth
of its subsentences

In the previous example we saw that, in the English language, the truth of a compound sentence depends on the truth of its subsentences. The same thing happens in mathematics, as discussed next.

S1 is sentence #1
S2 is sentence #2

Let $S1$ and $S2$ represent sentences that are either true or false. (For the moment, we don't want to consider sentences that are sometimes true, sometimes false.)

(Think of $S1$ as **S**entence $\#\mathbf{1}$, and $S2$ as **S**entence $\#\mathbf{2}$.)

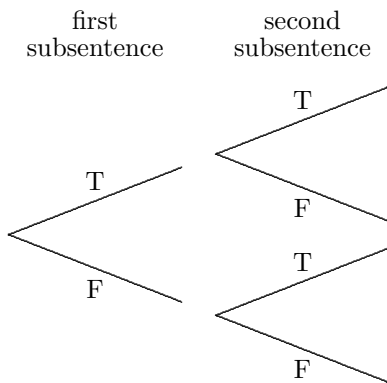Then, '$S1$ is equivalent to $S2$' is a compound sentence, built up from the subsentences $S1$ and $S2$:

(Note that a 'compound sentence' is still a *sentence*: it expresses a complete thought; it makes sense to ask about its truth.)

The truth of the compound sentence '$S1$ is equivalent to $S2$' depends on the truth of its subsentences, $S1$ and $S2$. Notice that $S1$ can be true or false; $S2$ can also be either true or false.
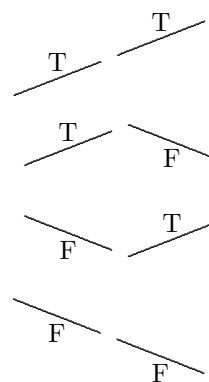
In mathematics, you are told how the truth of a compound sentence relates to the truth of its subsentences by a device called a **truth table**. A **truth table** is just a table that shows what happens for all possible combinations of truth values.

tree diagram
to summarize
all possible
truth combinations
for the subsentences

The first few columns in any truth table list all the possible truth combinations for the subsentences. Each subsentence can be either true (T) or false (F). The 'tree diagram' below (so-called, because it's supposed to remind you of a tree and its branches) shows that there are four possible truth combinations when there are two subsentences:

- top branch: first subsentence T, second subsentence T

- next branch: first subsentence T, second subsentence F

- next branch: first subsentence F, second subsentence T

- last branch: first subsentence F, second subsentence F

truth table for
the compound sentence
'S1 is equivalent to S2'

All that remains is to *define* the truth of the compound sentence for each of these four possible combinations.

So ... here's your first truth table:

DEFINITION
S1 is equivalent to S2

subsentence $S1$	subsentence $S2$	compound sentence $S1$ is equivalent to $S2$
T	T	T
T	F	F
F	T	F
F	F	T

What information
is the truth table
giving us?

Here's how to 'read' this truth table:

FIRST LINE: When $S1$ and $S2$ are both true, then the compound sentence '$S1$ is equivalent to $S2$' is also true.

SECOND LINE: When $S1$ is true and $S2$ is false, then the compound sentence '$S1$ is equivalent to $S2$' is false.

THIRD LINE: When $S1$ is false and $S2$ is true, then the compound sentence '$S1$ is equivalent to $S2$' is false.

FOURTH LINE: When $S1$ is false and $S2$ is false, then the compound sentence '$S1$ is equivalent to $S2$' is true.

*When is
the compound sentence
'S1 is equivalent to S2'
TRUE?*

So, when is the compound sentence 'S1 is equivalent to S2' true? Precisely when the subsentences S1 and S2 have the *same truth values*: when both subsentences are *true*; or when both subsentences are *false*.

*When is
the compound sentence
'S1 is equivalent to S2'
FALSE?*

When is the compound sentence 'S1 is equivalent to S2' false? Precisely when the subsentences S1 and S2 have *different truth values*: when one subsentence is *true*, but the other subsentence is *false*.

EXAMPLE

Decide whether the following compound sentences are true or false. State the line of the truth table being used to reach your decision.

1. $1 + 1 = 2$ is equivalent to $2 + 2 = 4$ _____
2. $1 + 1 = 2$ is equivalent to $2 + 2 = 5$ _____
3. $1 + 1 = 3$ is equivalent to $2 + 2 = 4$ _____
4. $1 + 1 = 3$ is equivalent to $2 + 2 = 5$ _____

SOLUTIONS:

1.

compound sentence is T

T   T

$1 + 1 = 2$ is equivalent to $2 + 2 = 4$

The compound sentence is *true*. Both subsentences have the same truth values: in this case, both subsentences are true. We're using line 1 of the truth table.

2.

compound sentence is F

T   F

$1 + 1 = 2$ is equivalent to $2 + 2 = 5$

The compound sentence is *false*, since the subsentences have different truth values. We're using line 2 of the truth table.

3.

compound sentence is F

F   T

$1 + 1 = 3$ is equivalent to $2 + 2 = 4$

The compound sentence is *false*, since the subsentences have different truth values. We're using line 3 of the truth table.

4.

compound sentence is T

F   F

$1 + 1 = 3$ is equivalent to $2 + 2 = 5$

The compound sentence is *true*. Both subsentences have the same truth values: in this case, both subsentences are false. We're using line 4 of the truth table.

This last example is most difficult for students: how can something be *true* with so much *false* stuff floating around? Remember: being 'equivalent' has to do with truth values being *the same*—either both true, *or* both false!

SUMMARY

In summary, it's convenient to remember that when the sentence 'S1 is equivalent to S2' is *true*, then you can make the following mental word replacement:

has the same truth value as

S1   is equivalent to   S2

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EXERCISES

6. Determine whether the following (compound) sentences are true or false. State which line of the truth table is being used.

(a) $2 + 2 = 4$ is equivalent to $3 + 6 = 9$

(b) $2 + 2 = 5$ is equivalent to $3 + 6 = 9$

(c) $2 + 2 = 5$ is equivalent to $3 + 6 = 10$

(d) $2 + 2 = 4$ is equivalent to $3 + 6 = 10$

7. Suppose you're told that the compound sentence '$S1$ is equivalent to $S2$' is TRUE. What (if anything) can be said about the truth values of the subsentences $S1$ and $S2$?

8. Suppose you're told that the compound sentence '$S1$ is equivalent to $S2$' is FALSE. What (if anything) can be said about the truth values of the subsentences $S1$ and $S2$?

alternate ways to say 'is equivalent to'

Equivalence is so extremely important in mathematics, that there are a variety of ways to say the same thing:

is equivalent to

if and only if

iff

$\iff$

In other words, even though the next four sentences *look* different, and are read aloud differently, they give precisely the same information:

$S1$ is equivalent to $S2$

$S1$ if and only if $S2$

$S1$ iff $S2$

$S1 \iff S2$

comments on the synonyms for 'is equivalent to':

Here are comments on each of these synonyms for 'is equivalent to':

if and only if

- 'if and only if': The word 'if' and the phrase 'only if' have precise mathematical meaning when each is used alone. (These meanings will *not* be discussed in this text). However, when put together with the word 'and' to form the phrase 'if and only if', then a synonym for 'is equivalent to' results. Writers sometimes choose the phrase 'if and only if' because it reads more smoothly in a given situation.

iff

- 'iff': This is *not* just a misspelling of 'if'. Instead, it is an abbreviation for 'if and only if'. Whenever you encounter the abbreviation 'iff', you should read it as 'if and only if'. (DON'T read 'iff' as 'if'; they have different meanings.)

$\iff$

- $\iff$: The symbols $\Leftarrow$ and $\Rightarrow$ have precise mathematical meaning when each is used alone (which will *not* be discussed in this text). However, when combined into the symbol $\iff$, then a synonym for 'is equivalent to' results. Whenever you encounter this symbol, you can read it as 'is equivalent to' or 'if and only if'—your choice.

general stylistic considerations: *text mode* *versus* *display mode*	Generally speaking, it's best to use words (like 'is equivalent to') in **text mode**; i.e., when you're in the midst of a paragraph of text. It's more acceptable to use symbols (for example, ' $\iff$ ') when information is being **displayed**, like this: For all real numbers x, $$\overbrace{2x - 3 = 0 \iff x = \frac{3}{2}}^{\text{this information is being displayed}}.$$ **Display mode** is used when you want a little extra attention to be given to some result; it is used for emphasis. In display mode, information is usually centered, with a bit of extra space above and below.

EXERCISE	9. Using the synonyms for 'is equivalent to', give three different ways that mathematicians might write the (true) sentence: $\qquad$ For all real numbers t, $\ t - 4 = 0\ $ is equivalent to $\ t = 4$. $\qquad$ (*) 10. What information is sentence (*) giving you? (That is, translate the sentence.)

Exercise 11 will provide practice with the ideas and notation being discussed, by asking you to compare the truth values of two sentences, and to summarize your results using correct notation. Study the next example carefully to prepare yourself for Exercise 11:

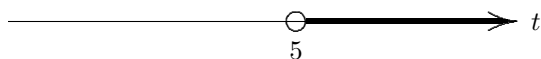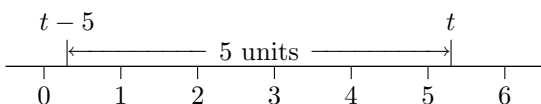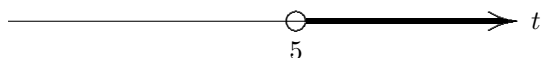|	
EXAMPLE	Consider this pair of sentences: $\qquad$ '$t > 5$' $\qquad\qquad$ and $\qquad\qquad$ '$t - 5 > 0$' First investigate '$t > 5$': What number(s) are greater than 5? Answer: All the numbers to the right of 5 on a number line. Thus, the sentence '$t > 5$' is true for the numbers shaded below, and false for those that are not shaded: Next investigate '$t - 5 > 0$': What number(s), when 5 is subtracted from them, yield a result that is greater than 0? Answer: All the numbers to the right of 5 on a number line. Thus, the sentence '$t - 5 > 0$' is true for the numbers shaded below, and false for those that are not shaded: Comparing results, we see that even though the sentences '$t > 5$' and '$t - 5 > 0$' *look* different, they *always have the same truth values*. They are true at the same time, and false at the same time. The sentence '$t > 5$' is easier to work with, because it 'tells' you when it is true.

summarizing
the observations

These observations could be summarized by stating any one of the following:

For all real numbers t, $t > 5$ is equivalent to $t - 5 > 0$.

For all real numbers t, $t > 5$ if and only if $t - 5 > 0$.

For all real numbers t, $t > 5$ iff $t - 5 > 0$.

For all real numbers t, $t > 5 \iff t - 5 > 0$.

EXERCISE

11. For each pair of sentences below:

Determine when the first sentence is true, and when it is false.

Determine when the second sentence is true, and when it is false.

Compare your results: Do the two sentences always have the same truth values?

Is one sentence easier to work with than the other?

Give four different ways that you might summarize your observations.

(a) $5 - t = 0$, $t = 5$

(b) $3x = 12$, $x = 4$

(c) $x = 14$, $\frac{x}{2} = 7$

---

two more
sentence connectives:
the mathematical words
'and' and 'or'

Next, two more mathematical sentence connectives are investigated:

'AND' and 'OR'.

The words 'and' and 'or' are used in both English and mathematics. We've already seen that the *English* word 'and' can be used as either a sentence connective, or as a connective for nouns.

In *mathematics*, the words 'and' and 'or' are used *only as sentence connectives*. Given sentences $S1$ and $S2$, we can form compound sentences:

'$S1$ and $S2$'

'$S1$ or $S2$'

The truth of each compound sentence depends on the truth of its subsentences.

The truth table for 'and' and 'or' sentences is given below:

DEFINITIONS

S1 and S2 ;

S1 or S2

$S1$	$S2$	$S1$ and $S2$	$S1$ or $S2$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

---

mathematical
'and' sentence:
when is it true?
False?

By studying the truth table, you will observe that:

• The only time an 'and' sentence is true is when both subsentences are true (line 1 of the truth table).

An 'and' sentence is false when at least one of the subsentences is false. In other words, an 'and' sentence is false when one, or the other, or both, of the subsentences are false (lines 2, 3, and 4 of the truth table).

This is precisely in agreement with the English usage of the word 'and'. Indeed, the author has used the mathematical word 'and' several times in this book, and was able to get away with it because the English and mathematical meanings (as sentence connectives) agree so well.

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*mathematical
'or' sentence:
when is it true?
False?*

Again by studying the truth table, you will observe that:

- An 'or' sentence is true when at least one of the subsentences is true. In other words, an 'or' sentence is true when one, or the other, or both, of the subsentences are true (lines 1, 2, and 3 of the truth table).

  The only time an 'or' sentence is false is when both of the subsentences are false (line 4 of the truth table).

*'or' is used
a bit differently
in mathematics
and English*

The mathematical 'or' is a bit different from the typical English usage of the word 'or', and hence deserves some comment.

If you say to a friend,

"I'm going to get cheesecake, or I'm going to get apple pie."

then you probably mean that you'll get one or the other, but *not both*.

However, the *mathematical* sentence '$S1$ or $S2$' is indeed true when both $S1$ and $S2$ are true (line 1). Be careful about this!

The next example provides practice with 'and' and 'or' sentences.

EXAMPLE
*truth of
'and' and 'or'
sentences*

Decide whether the following sentences are *true* or *false*. In each case, state the line of the truth table that you are using.

1. $1 + 1 = 2$ and $3 + 3 = 6$ _____
2. $1 + 1 = 2$ and $3 + 3 = 7$ _____
3. $1 + 1 = 2$ or $3 + 3 = 7$ _____
4. $1 + 1 = 2$ or $3 + 3 = 6$ _____

SOLUTIONS:

SOLUTIONS:

1. $\overbrace{\underbrace{1+1=2}_{T} \text{ and } \underbrace{3+3=6}_{T}}^{T}$

   An 'and' sentence is true when both subsentences are true. (line 1)

2. $\overbrace{\underbrace{1+1=2}_{T} \text{ and } \underbrace{3+3=7}_{F}}^{F}$

   An 'and' sentence is false when at least one subsentence is false. (line 2)

3. $\overbrace{\underbrace{1+1=2}_{T} \text{ or } \underbrace{3+3=7}_{F}}^{T}$

   An 'or' sentence is true when at least one subsentence is true. (line 2)

4. $\overbrace{\underbrace{1+1=2}_{T} \text{ or } \underbrace{3+3=6}_{T}}^{T}$

   An 'or' sentence is true when *at least one* subsentence is true. That is, exactly one of the subsentences may be true, or both of the subsentences may be true. In this case, both of the subsentences are true, so the compound sentence is true. (line 1)

SUMMARY

In summary:

'$S1$ and $S2$' is true only when both $S1$ and $S2$ are true

'$S1$ or $S2$' is true when $S1$ is true, or $S2$ is true, or both are true

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EXERCISES

12. Determine the truth value (T or F) of each of the following compound sentences. In each case, state the line of the truth table that is being used.

(a) $1 + 1 = 3$ and $2 + 2 = 5$

(b) $1 + 1 = 3$ and $2 + 2 = 4$

(c) $1 + 1 = 3$ or $2 + 2 = 4$

(d) $1 + 1 = 3$ or $2 + 2 = 5$

13. Suppose you are told that the sentence '$S1$ and $S2$' is true. What (if anything) can be said about the truth values of the subsentences $S1$ and $S2$?

14. Suppose you are told that the sentence '$S1$ and $S2$' is false. What (if anything) can be said about the truth values of the subsentences $S1$ and $S2$?

15. Suppose you are told that the sentence '$S1$ or $S2$' is true. What (if anything) can be said about the truth values of the subsentences $S1$ and $S2$?

16. Suppose you are told that the sentence '$S1$ or $S2$' is false. What (if anything) can be said about the truth values of the subsentences $S1$ and $S2$?

17. Suppose you want to know the truth of the compound sentence '$S1$ or $S2$', and you've already determined that $S1$ is true. Do you need to investigate $S2$?

18. Suppose you want to know the truth of the compound sentence '$S1$ and $S2$', and you've already determined that $S1$ is false. Do you need to investigate $S2$?

---

compound sentences using variables

Recall a fact about variables: *whenever you see a variable in mathematics, there's a universal set for that variable.* Sometimes you are told precisely what the universal set is; other times, it's just left 'lurking in the background'.

In the sentence

$\qquad$ 'For all real numbers x, $2x - 3 = 0$ is equivalent to $x = \frac{3}{2}$.' $\qquad$ (*)

the universal set for x has been explicitly stated: it is the set of all real numbers. You are being told that no matter what real number is substituted for x,

$\qquad$ '$2x - 3 = 0$' has the same truth value as '$x = \frac{3}{2}$'.

Explicitly stating the universal set is, indeed, the *strictly correct* way to give the information.

However, when you communicate with someone regularly, then you can often give an 'incomplete' request, and get back precisely what you want. Or, you can make a somewhat 'incomplete' statement, and have it interpreted completely correctly. The same is true in mathematics.

In the spirit of 'You know what I mean anyway . . . ', people writing mathematics occasionally get a bit lazy. Sometimes, the 'for all' part of sentence (*) is dropped in casual conversation, and you just see (or hear):

$\qquad$ '$2x - 3 = 0$ is equivalent to $x = \frac{3}{2}$'

Notice that some important information has been omitted: for *which* values of x do the two sentences have the same truth values?

The 'agreement' among mathematicians is that the 'for all' part is only omitted in casual situations, and it is only omitted if the two sentences being compared *always* have the same truth values.

When is it okay to omit the 'for all'? *Three conditions must be met:*	More precisely, when people leave off the 'for all' information in comparing two mathematical sentences, then they customarily mean that *the following three things are true.* (The first two conditions are discussed in more detail momentarily.)

(1) **The two sentences use exactly the same variable(s).**

(2) **The two sentences are nonsensical for exactly the same numbers (if at all).**

(3) **The two sentences have** *precisely* **the same truth values for** *all* **the variables for which they are (mutually) defined.**

The first two conditions deserve some additional explanation:

(1) sentences use the same variables

We want the sentences being compared to use precisely the same variable(s). Perhaps they both use the variable x; or perhaps they both use the variable t. Or, perhaps, they both use the variables x and y. So, feel free to compare the sentences '$x = 3$' and '$x - 3 = 0$'; but please don't compare '$x = 3$' (which uses the variable x) with '$t - 3 = 0$' (which uses the variable t).

(2) sentences are nonsensical at the same time:

Some sentences just *don't make sense* for a particular choice of variable. For example, you can't substitute zero for x in the sentence '$\frac{1}{x} = 3$', because division by zero isn't allowed.

defined *versus* **not defined**

We want the sentences that are being compared to 'not make sense' at exactly the same times. That is, if one sentence is nonsensical for a particular choice of variable(s), then the other sentence should also be nonsensical for this same choice of variable(s).

The most common 'nonsensical' situation that arises is division by zero. Division by zero just isn't allowed; it doesn't make sense; it's nonsense. For example, you're not allowed to substitute 5 in for x in the sentence '$\frac{1}{x-5} = 2$' because this would cause division by zero.

In mathematics, we say that a sentence **IS NOT defined** when nonsense results. For example, the sentence '$\frac{1}{x-5} = 2$' **IS NOT defined** when x is 5.

In mathematics, we say that a sentence **IS defined** when either a true or false sentence results. For example, the sentence '$\frac{1}{x-5} = 2$' **IS defined** whenever x is not 5; substitution of any number other than 5 for x results in a sentence that is either true or false.

BE CAREFUL! Being **defined** and being **true** are totally different ideas! For example, the sentence '$\frac{1}{x-5} = 2$' IS defined when x is zero. It just happens to be false when x is zero.

sentences that are completely interchangeable

When ALL THREE of the conditions just discussed are met, then two sentences are indeed *completely interchangeable*. They use the same variable(s). They're true at the same time. They're false at the same time. They're not defined at the same time. So, you can feel free to drop the 'for all' in such situations. The following examples provide some practice with these ideas.

EXAMPLE

Let's compare the sentences '$\frac{1}{x} = \frac{1}{3}$' and '$\frac{3-x}{3x} = 0$'.

Both sentences use the same variable, x.

Both sentences are nonsensical only when $x = 0$.

Both sentences are true when $x = 3$, and false for all other nonzero numbers.

Thus, in casual conversation, feel free to say:

$$\text{'}\frac{1}{x} = \frac{1}{3} \text{ is equivalent to } \frac{3-x}{3x} = 0\text{'}$$

instead of the more correct statement

$$\text{'For all nonzero real numbers } x, \ \ \frac{1}{x} = \frac{1}{3} \ \text{ is equivalent to } \ \frac{3-x}{3x} = 0\text{.'}$$

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EXAMPLE

Next, let's compare the sentences ' $\frac{1}{x} = \frac{1}{3}$ ' and ' $x = 3$ '.

Both sentences use the same variable, x .

The first sentence is nonsensical when $x = 0$. However, the second sentence is defined for all real numbers x .

Both sentences are true when $x = 3$.

Both sentences are false when $x \neq 0$ and $x \neq 3$.

Notice that if we choose x to be 0 , then the first sentence is *not* defined, but the second sentence *is* defined (and is false).

Because of this discrepancy, it's probably best *not* to drop the 'for all', and to state things correctly, by saying:

'For all nonzero real numbers x , $\frac{1}{x} = \frac{1}{3}$ is equivalent to $x = 3$.'

Indeed, one sign of mathematical maturity is having the ability to *recognize* when something is not being stated completely correctly; and having the ability to correct the problem, if so asked!

END-OF-SECTION EXERCISES

For problems 19–30: Classify each entry as a mathematical expression (EXP), or a mathematical sentence (SEN).

Classify the truth value of any entry that is a sentence: (always) true (T); (always) false (F); or sometimes true/sometimes false (ST/SF).

19. $2x - 3$
20. $2x - 3 = 0$
21. $x = \frac{3}{2}$
22. $2(\frac{3}{2}) - 3 = 0 \iff \frac{3}{2} = \frac{3}{2}$
23. $2(5) - 3 = 0 \iff 5 = \frac{3}{2}$
24. $2x - 3 = 0 \iff x = \frac{3}{2}$
25. For all real numbers x , $2x - 3 = 0$ iff $x = \frac{3}{2}$.
26. $2(\frac{3}{2}) - 3 = 0$ and $\frac{3}{2} = \frac{3}{2}$
27. $2(\frac{3}{2}) - 3 = 0$ or $\frac{3}{2} = \frac{3}{2}$
28. $2(5) - 3 = 0$ and $\frac{3}{2} = \frac{3}{2}$
29. For all real numbers t , t is greater than 5 if and only if 5 is less than t .
30. For all real numbers x , $x > 5$ is equivalent to $5 < x$.

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SECTION SUMMARY
THESE SENTENCES CERTAINLY LOOK DIFFERENT

NEW IN THIS SECTION	HOW TO READ	MEANING
equality of expressions		In general, mathematicians talk about *expressions being equal*. *Numbers* are equal when they live at the same place on a number line. *Sets* are equal when they have precisely the same members.
equivalence of sentences		In general, mathematicians talk about *sentences being equivalent*. This has to do with the sentences having the same truth values. (The precise definition appears below; also see the discussion on pages 105–107.)
connective		A *connective* is used to connect two objects of the same type into a compound object of the same type. Numbers can be 'connected' to get a new number; sets can be 'connected' to get a new set; sentences can be 'connected' to get a new sentence.
connectives for numbers		The four most common connectives for numbers are: $+$ (addition) $-$ (subtraction) $\cdot$ (multiplication) $/$ (division)
truth table		A truth table is a table that shows you how the truth values of a compound sentence relate to the truth values of its subsentences.
DEFINITION: '$S1$ is equivalent to $S2$'		 $\begin{array}{ccc} S1 & S2 & S1 \text{ is equivalent to } S2 \\ \\ T & T & T \\ T & F & F \\ F & T & F \\ F & F & T \end{array}$ Two sentences are equivalent when they have the same truth values.
is equivalent to if and only if iff $\Longleftrightarrow$	(see individual entries)	The idea of mathematical equivalence is so important, that there are a variety of ways to say the same thing. These are all synonyms.

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NEW IN THIS SECTION	HOW TO READ	MEANING
iff	'if and only if'	a synonym for 'is equivalent to'; an abbreviation for 'if and only if'
$\Longleftrightarrow$	'is equivalent to' or 'if and only if'	A synonym for 'is equivalent to'. This symbol is often used in display mode.
DEFINITION: '$S1$ and $S2$' (the mathematical word 'and')		The mathematical word 'and' is a sentence connective, defined by the truth table: $S1$ $S2$ $S1$ and $S2$ T T T T F F F T F F F F An 'and' sentence is true only when both subsentences are true.
DEFINITION: '$S1$ or $S2$' (the mathematical word 'or')		The mathematical word 'or' is a sentence connective, defined by the truth table: $S1$ $S2$ $S1$ or $S2$ T T T T F T F T T F F F An 'or' sentence is true when at least one of the subsentences is true. That is, an 'or' sentence is true when one, or the other, or both, of the subsentences are true. Notice that line 1 of the truth table is slightly different than the English word 'or'.
not defined		Mathematicians say that a sentence **is not defined** when nonsense results. For example, the sentence '$\frac{1}{x-5} = 2$' **is not defined** when x is 5.
defined		Mathematicians say that a sentence **is defined** when either a true or false sentence results. For example, the sentence '$\frac{1}{x-5} = 2$' **is defined** whenever $x \neq 5$: substitution of any number other than 5 for x results in a sentence that is either true or false.

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