2. THE REAL NUMBERS

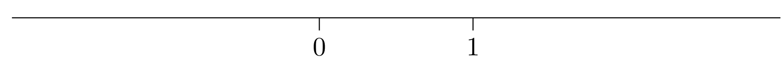
numbers; an introduction

Numbers are used in a variety of ways:

- to count things: e.g., 3 books
- to measure things: e.g., $\frac{1}{2}$ cup milk
- to identify things: e.g., stock #1730412
- to order things: 1^{st}, 2^{nd}, 3^{rd}, ...

Frequently encountered are *decimals* (like \$3.25), *fractions* (like $\frac{3}{4}$ cup), and *percents* (like 5% annual interest rate).

Although numbers come in lots of different sizes, and have lots of different names, here's the good news: *all these numbers live on the 'line' shown below*. This is called a *real number line*, and is the subject of this section.



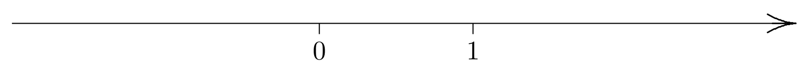
a real number line

A *real number line* is determined by three pieces of information:

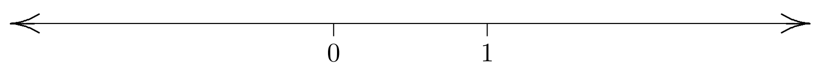
- a (straight) line;

  (Even though this line may have any orientation, the following discussion assumes that the line is horizontal.)
- a point on the line, usually labeled as the number 0 (zero); and
- a second point on the line, to the right of the first point, usually labeled as the number 1 (one).

Sometimes, an arrow is put at the right end of a number line, to show that this is the 'positive' direction; that is, the numbers increase as you move to the right.



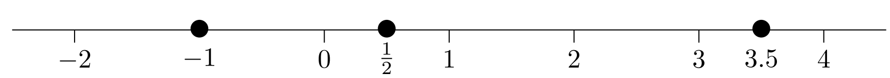
Sometimes, arrows are put at both ends, to suggest that the line extends forever in both directions.



Sometimes, there are no arrows at all: this is the simplest representation, and is the one that will be used most often in this book.

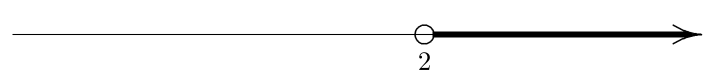
with choices made for 0 and 1, the locations of all other numbers are uniquely determined

With choices made for 0 and 1, the locations of *all other numbers are uniquely determined*: for example, the locations of -1 (negative one), $\frac{1}{2}$ (one-half), and 3.5 ('three point five' or 'three and five-tenths') are shown below:



a number line when only a single number is of interest

Even though *two different numbers* are required to determine where all the other numbers live, people occasionally get lazy. If there's only a *single* number that is currently of interest, then a 'number line' may be drawn showing *only* that particular number. For example, all numbers to the right of 2 might be illustrated like this:



(The hollow dot at 2 indicates that 2 is *not* to be included.)

a real number line is a conceptually perfect picture of the real numbers

A number line provides us with a *picture* of a collection of numbers referred to as the *real numbers*. It is a conceptually perfect picture, in the following sense:

- every point on the line corresponds to a real number; and
- every real number corresponds to a point on the line.

Since there is this 'pairing' of real numbers and points on the line, people tend to use the words 'number' and 'point' interchangeably. So will this author.

the symbol $\mathbb{R}$

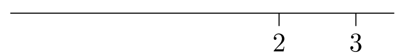
The collection of real numbers is denoted by the symbol $\mathbb{R}$. Whenever you encounter this symbol, it's most correct to read it as *the real numbers*. (However, people often get lazy and read the letter literally, as 'arr'.)

The idea of 'collection' is made precise in the next section, **Mathematicians are Fond of Collections**, where *sets* are discussed.

EXERCISES

1. Draw a number line, where the distance from 0 to 1 is one inch. Then, locate the following numbers: 2, $\frac{1}{3}$, $\frac{1}{4}$, and -2.
2. Repeat exercise 1, but this time on a number line where the distance from 0 to 1 is one-half inch.
3. Although it is conventional to use 0 and 1 when creating a number line, *any* two different numbers can be used to determine the locations of all other numbers. To explore this idea, consider the number line below, where the positions of 2 and 3 have been specified.

   Determine the locations of 0 and 1.

   Then locate -1, $1\frac{1}{2}$, and $2\frac{1}{4}$.

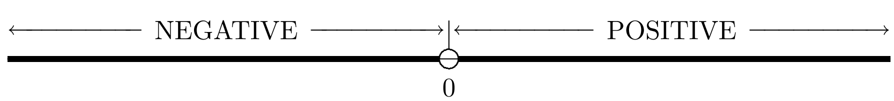
★
distinction between $\mathbb{R}$ and the 'picture' representing $\mathbb{R}$

Occasionally, the author will put some information in a box that is labeled with a ★. Such information is included for the benefit of readers with considerable mathematical experience. This device allows the author to state more of the complete truth, without interrupting the exposition. Most readers will SKIP all ★ sections (at least on a first reading) without any loss of continuity.

There is a unique set of real numbers, $\mathbb{R}$. However, there are an infinite number of choices for our *representation* of this set, corresponding to the choices for 0 and 1. Thus, the set of real numbers is unique (hence the phrase *'the' real numbers*) but the representation is not unique (hence the phrase *'a' real number line*).

positive and negative real numbers

The numbers to the right of zero are called the *positive* (POS–i–tiv) real numbers; the numbers to the left of zero are the *negative* (NEG–a–tiv) real numbers. The number zero is not positive (since it doesn't lie to the right of zero), and not negative (since it doesn't lie to the left of zero). Zero is the only real number with this 'neutral' status; every other real number is either positive or negative.



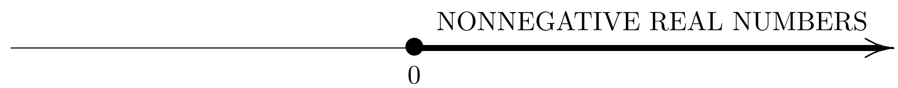
distinguish between the number 0 (zero) and the letter O (oh)

Be sure to read the number 0 as 'zero', and the letter O as 'oh'. Even though the symbols look almost identical when hand-written, context will usually tell you whether the symbol represents a number or a letter.

In computer science, where the difference between 0 and O becomes critical (due to the non-forgiving nature of computers), the symbol $\emptyset$ is often used for the number zero to prevent any confusion.

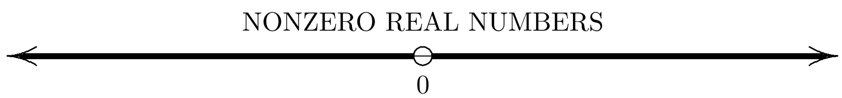
nonnegative real numbers

Which real numbers are *not* negative? Zero isn't negative. Also, the positive numbers are not negative. These numbers—zero, together with all real numbers to the right of zero—are called the *nonnegative* real numbers, and are shaded below. The solid (filled-in) dot at 0 indicates that 0 is being included; the arrow to the right indicates that the shading is to continue for all numbers to the right of zero.



nonzero real numbers

A *nonzero* real number is one that is not zero; the nonzero real numbers are shaded below. The hollow (not filled-in) dot at 0 indicates that 0 is NOT being included.



EXERCISE	4. Clearly shade the *nonpositive* real numbers on a number line. Is 0 included or not included?

whole numbers;

use of '. . . ' for continuing an established pattern

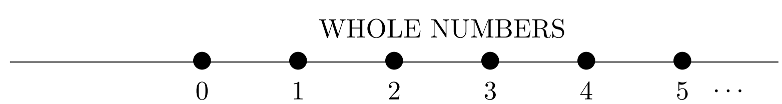
There are some important subcollections of $\mathbb{R}$ that are given special names. (Think of a 'subcollection' as 'part of' a collection.)

The *whole numbers* are the subcollection containing:

$$0\,,\ 1\,,\ 2\,,\ 3\,,\ \ldots$$

The three lower dots '. . . ' indicate that the established pattern is to be repeated ad infinitum (pronounced 'odd in–fi–NIGHT–um' or 'add in–fi–NIGHT–um'; means for ever and ever). Thus, 127 is a whole number, but $\frac{1}{2}$ isn't.

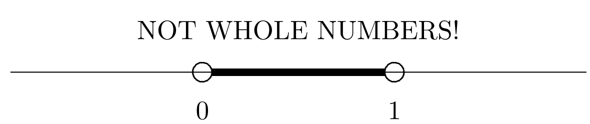
Most people read the list '$0, 1, 2, 3, \ldots$' as *'zero, one, two, three, dot, dot, dot'* or *'zero, one, two, three, and so on'*.



consecutive whole numbers

Consecutive whole numbers are whole numbers that follow one after the other, without gaps. The phrase can refer to just two numbers, or more than two. Thus, 2 and 3 are consecutive whole numbers; 5 , 6 , and 7 are consecutive whole numbers; 2 and 5 are *not* consecutive whole numbers.

Notice how sparse the whole numbers are, as they sit in the collection of real numbers! Between any two consecutive whole numbers are an infinite (IN–fi–nit) number of real numbers, that are NOT whole numbers.



EXERCISES

5. How would you read the sentence: *There are some important subcollections of* $\mathbb{R}$ *that are ...* ? That is, how do you read the symbol $\mathbb{R}$?
6. List four consecutive whole numbers, beginning with 7.
7. a) Which whole numbers are positive?
   b) Is there any whole number that is not positive?
   c) Which whole numbers are nonnegative?

size versus order:
size:
bigger, smaller

order:
greater than, less than

There are two different concepts frequently used to compare numbers:

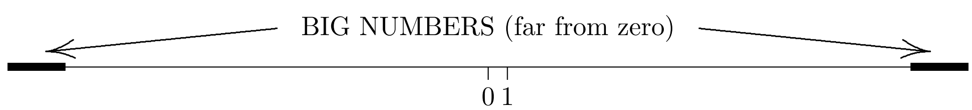
- SIZE: the *size* of a number refers to its distance from zero. The words 'bigger' and 'smaller' are used to talk about size.
- ORDER: there is a natural left/right ordering on the number line. Given any two numbers, either they are equal, or one lies to the right of the other. The words 'greater than' and 'less than' are used to talk about order.

The *size* of real numbers is discussed next; *order* will be discussed in the section **I Live Two Blocks West Of You**.

distance from zero:
big numbers
versus
small numbers

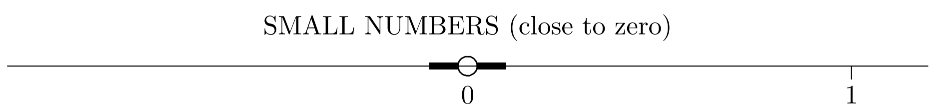
As mentioned above, the *size* of a number is given by its distance from zero.

Roughly, a number is 'big' or 'large' if it is far from zero:



Note that large numbers can be positive (like 1,000,000 ; one million) or negative (like -1,000,000 ; negative one million).

Roughly, a number is 'small' if it is close to zero:



Note that small numbers can be positive (like $\frac{1}{1000}$; one thousandth) or negative (like $-\frac{1}{1000}$; negative one thousandth). Usually, 'small' means close to zero, but not equal to zero.

EXERCISE

8. Using examples that are bigger/smaller than those cited in the book, give an example of a small positive number; a large positive number; a small negative number; a big negative number.

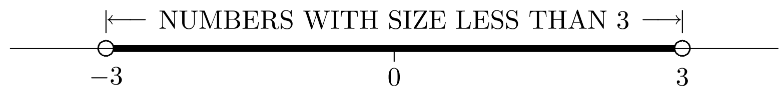
'size' is a
nonnegative quantity

Observe that when you report a number's distance from zero, you are always reporting a number that is nonnegative. (It doesn't make sense to say 'this number lives -2 units from zero'.) Since *size* gives a number's distance from zero, it follows that *size* is a nonnegative quantity. For example, both 5 and -5 are five units from zero. Thus, 5 and -5 have the same size: five.

EXERCISES

9. You might be used to measuring reading speed in units of *pages per hour*; however, reading *mathematics* is often measured in units of *hours per page*! You may need to read the previous paragraph several times before you understand it completely. Be sure that you can answer the following questions: What are the nonnegative numbers? What is the meaning of the sentence that begins with 'Since *size* gives a number's ...' ?
10. Give the size of each real number: 2 , -2 , 0 , 10.2 , -10.2 .

Suppose you are asked to shade all real numbers that are less than 3 units from zero. The key idea is this: you can move away from zero in *two different directions*. You could move less than three units *to the right*; or, you could move less than three units *to the left*. The resulting numbers, shaded below, all have size less than 3 .



★ *absolute value*

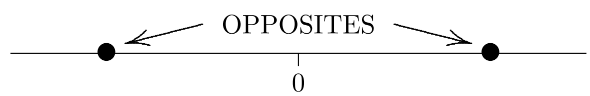
The absolute value of a number x , denoted by $|x|$, gives its distance from zero on a number line. That is, absolute value measures the *size* of a number. Thus, $|5| = 5$ and $|-5| = 5$.

EXERCISES

11. Draw a number line, and shade all real numbers that are exactly two units from zero.
12. Draw a number line, and shade all real numbers that are less than two units from zero. Be sure to clearly label any endpoint(s) with solid or hollow dots (whichever is appropriate).
13. Draw a number line, and shade all real numbers that are more than two units from zero. Be sure to clearly label any endpoint(s) with solid or hollow dots (whichever is appropriate).
14. Draw a number line, and shade all real numbers with size less than or equal to 2 . Be sure to clearly label any endpoint(s) with solid or hollow dots (whichever is appropriate).
15. Is there a largest whole number? Explain.

opposites

Numbers like 2 and −2 are called *opposites* (OPP–po–sits): they have the same distance from zero, but are on opposite sides of zero. The opposite of a positive number is a negative number. The opposite of a negative number is a positive number. The opposite of zero is zero: zero is the only real number that is its own opposite.



adding a number to its opposite

Whenever you add a number to its opposite, you get zero as a result:

$$3 + (-3) = 0$$
$$(-2) + 2 = 0$$
$$5.1 + (-5.1) = 0$$
$$0 + 0 = 0$$

how to read something like '-3'

The number −3 can be read as either 'negative three' or 'the opposite of three'. Many people favor 'negative three', because it's faster. However, both ways are correct.

Similarly, $-x$ can be read as either 'negative ex' or 'the opposite of ex'. Here, the letter x is being used to represent a number: such use of letters to represent numbers is discussed in the section **Holding This, Holding That**. We'll see in this future section why it is preferable for beginning students of mathematics to read $-x$ as 'the opposite of x'.

★

Some people read '$-x$' as 'minus x'. This author, however, prefers to reserve the word 'minus' for the operation of subtraction.

EXERCISES

16. What is the opposite of $\frac{1}{2}$? What is the opposite of $-\frac{1}{2}$?
17. I'm thinking of a number that lies two units from zero on a number line. What number(s) could I be thinking of?
18. I'm thinking of a number that has size 3. What number(s) could I be thinking of?

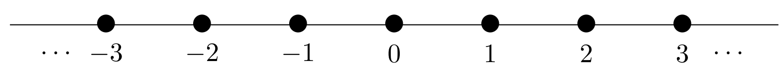
integers, $\mathbb{Z}$

When we take the whole numbers, and throw in their opposites, then we get the important subcollection of $\mathbb{R}$ called the *integers* (IN–teh–jers). Thus, the integers are the subcollection:

$$\ldots\,,\ -3\,,\ -2\,,\ -1\,,\ 0\,,\ 1\,,\ 2\,,\ 3\,,\ \ldots$$

The symbol $\mathbb{Z}$ is used to represent the integers (from the German 'Zahlen', meaning 'numbers'). Both 127 and -127 are integers; $\frac{1}{2}$ is not an integer.

INTEGERS



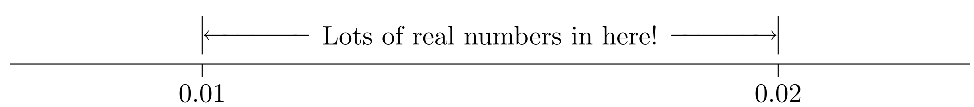
EXERCISES

19. Give an example of an integer that is not a whole number.
20. What are the positive integers? The negative integers?
21. What are the nonnegative integers? The nonpositive integers?
22. Give an example of a large negative integer; a large positive integer.
23. a) I'm thinking of an integer. Is its opposite an integer? Explain.
    b) I'm thinking of a number that is NOT an integer. Is its opposite an integer? Explain.

density property of the real numbers

One important property of the real numbers is that they are *dense*; that is, between every two different real numbers (no matter how close they are), there is another real number. Indeed, between every two different real numbers, there are an infinite number of real numbers!

DENSITY PROPERTY OF THE REAL NUMBERS



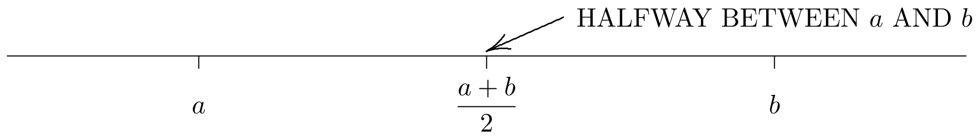
averaging two numbers

It's easy to find a number that lies between two given numbers: just average them. To *average two numbers* means to add the numbers together, and then divide by 2 . Thus,

$$\text{the average of } a \text{ and } b \text{ is } \frac{a+b}{2}\ .$$

averaging gives the midpoint

Averaging two different numbers always yields the number exactly halfway between, as illustrated below.



For example, averaging 2 and 4 gives $\dfrac{2+4}{2} = \dfrac{6}{2} = 3$; averaging 0.1 and 0.2 gives $\dfrac{0.1+0.2}{2} = 0.15$.

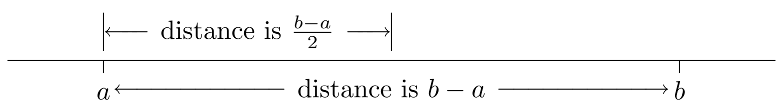
How do we know that $\frac{a+b}{2}$ really lies halfway between a and b?

Clearly, the formula $\frac{a+b}{2}$ gives *some* number; but how do we know that the number given by this formula is *really*, *always*, halfway between a and b? Although repeated trials (with lots of different numbers) is pretty convincing, it is of course impossible to check *every* pair of real numbers. Thus, mathematics-type people prefer to prove their point with an argument like the one shown below. (This argument can be skipped without any loss of continuity.)

★ *algebraic proof for more experienced readers; the average of two numbers gives a number that is exactly halfway between*

Let a and b be different real numbers; rename, if necessary, so that $a < b$. The distance between a and b is $b - a$, and half this distance is $\frac{b-a}{2}$. Then, the midpoint of a and b is:

$$a + \frac{b-a}{2} = \frac{2a}{2} + \frac{b-a}{2} = \frac{2a+b-a}{2} = \frac{a+b}{2}\,.$$



EXERCISES

24. a) Find the average of 2 and 6.
    b) Find the number exactly halfway between 2 and 6 on a number line. Compare with a).
25. The numbers 0.13 and 0.14 are very close to each other. Find a number halfway between them. (Use a calculator, if necessary.)
26. What happens if you average two numbers that are the same?

equality of real numbers

If two numbers live at the same place on a real number line, then we say that they are *equal*. And, if two numbers are *equal*, this means that they live at the same place on a real number line.

the mathematical sentence '$a = b$'

The mathematical sentence '$a = b$' is read as 'a equals b' or 'a is equal to b'. This sentence is TRUE if a and b live at the same place on a real number line; otherwise, it's false. Note that if the sentence '$a = b$' is TRUE, then you're being told that 'a' and 'b' are just different names for the same number!

the mathematical sentence '$a \ne b$'

The mathematical sentence '$a \ne b$' is read as 'a does not equal b' or 'a is not equal to b'. This sentence is TRUE when a and b live at different places on a real number line; otherwise, it's false.

EXERCISES

27. On a number line, label points a and b reflecting each situation described below:
    a) The sentence $a = b$ is true.
    b) The sentence $a = b$ is false.
    c) The sentence $a \neq b$ is true.
    d) The sentence $a \neq b$ is false.

'number' means 'real number'

Throughout this book, when the word 'number' is used, it means 'real number'. Thus, if you're asked:

What are the positive numbers?

you are to assume that you're being asked for the positive *real* numbers, and must respond by shading:

Notice that you *can't list* the positive real numbers—you *must* shade them on a number line!

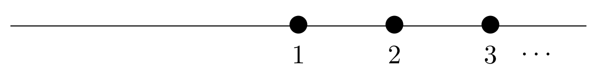
However, if you're asked:

What are the positive integers?

then you could certainly respond with a list

$$1\,,\ 2\,,\ 3\,,\ \ldots$$

OR with a picture



END-OF-SECTION EXERCISES

For problems 28–34: Classify each entry as a mathematical expression (EXP), or a mathematical sentence (SEN).

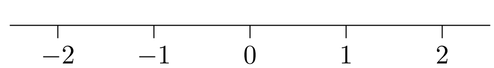
In each sentence, circle the verb.

Classify the truth value of each entry that is a sentence: (always) true (T); (always) false (F); or sometimes true/sometimes false (ST/SF). The first one is done for you.

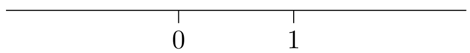
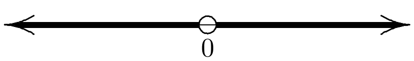
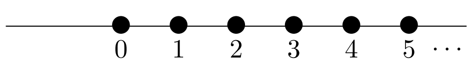
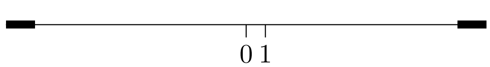
	3 is a positive number	SEN, T
28.	0 is a nonnegative number	__________
29.	x is a positive number	__________
30.	The numbers 2 and $1+1$ are equal.	__________
31.	$x = y$	__________
32.	$x + y$	__________
33.	Every whole number is an integer.	__________
34.	Every integer is a whole number.	__________

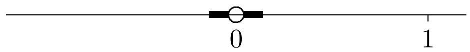
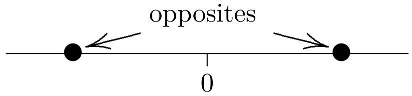
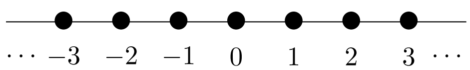
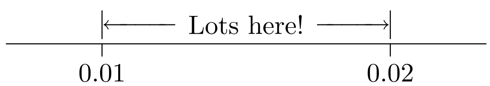
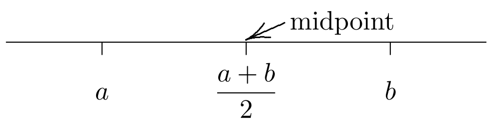
Solve this additional problem:

35. On the number line below, shade the numbers between -1 and 2 (not including the endpoints).



SECTION SUMMARY
THE REAL NUMBERS

NEW IN THIS SECTION	HOW TO READ	MEANING
uses for numbers		to count things; to measure things; to identify things; to order things
$\mathbb{R}$	the real numbers	
real number line 		a conceptually perfect picture of the real numbers
positive 	POS–i–tiv	to the right of zero on a number line
negative 	NEG–a–tiv	to the left of zero on a number line
0 or $\emptyset$	zero (ZEE–row)	When confusion with the capital letter O could result, use $\emptyset$ to represent the number zero.
nonnegative 	non–NEG–a–tiv	not negative: positive or zero
nonzero 	non–ZEE–row or NON–zee–row	not equal to zero
solid dot ●		indicates that a number is included
hollow dot ○		indicates that a number is not included
whole numbers 		the collection: $0\,,\ 1\,,\ 2\,,\ 3\,,\ \ldots$
consecutive		following one after the other, without gaps
size of a number		The *size* of a number refers to the number's distance from zero on a number line.
order		*Order* refers to a natural left/right ordering on a number line. Given any two numbers, either they are equal, or one lies to the right of the other.
big or large 		far away from zero on a number line

NEW IN THIS SECTION	HOW TO READ	MEANING
small 		close to zero (but not equal to zero) on a number line
opposites 	OPP–po–sits	numbers that are the same distance from zero, but on opposite sides of zero (like 2 and -2); zero is its own opposite
-3	'negative three' or 'the opposite of three'	
$\mathbb{Z}$ 	the integers (IN–teh–jers)	the collection: $\ldots\,,\ -3, -2, -1, 0, 1, 2, 3,\ \ldots$
density property of $\mathbb{R}$ 		Between every two different real numbers (no matter how close), there is another real number.
average of a and b 		The average of a and b is found by adding the numbers, and then dividing by 2: $$\frac{a+b}{2}$$ The average of a and b lies halfway between a and b.
$a = b$	'a equals b' or 'a is equal to b'	A mathematical sentence: true, when a and b live at the same place on a real number line; false, otherwise.
$a \neq b$	'a does not equal b' or 'a is not equal to b'	A mathematical sentence: true, when a and b live at different places on a real number line; false, otherwise.
number		In this book, *number*, unless otherwise specified, refers to a REAL number.