LINEAR INEQUALITIES IN TWO VARIABLES

- Need some basic understanding of sentences in two variables first?
  Introduction to Equations and Inequalities in Two Variables

(more mathematical cats)

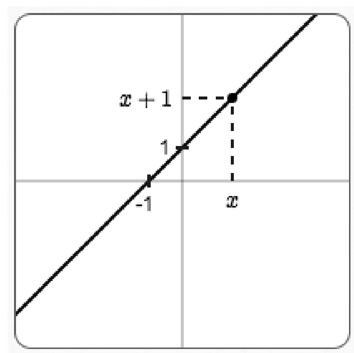
Here's what this lesson offers:

- Going from Linear Equations to Linear Inequalities: the graphs change dramatically!
- Important Concepts for Graphing Linear Inequalities in Two Variables
- The Test Point Method for Graphing Linear Inequalities in Two Variables
- Special Linear Inequalities in Two Variables: you only ***see*** one variable

Going from Linear Equations to Linear Inequalities

You've already learned that the graph of $y = x + 1$ is the line shown at right.

This line is the picture of all the points (x, y) that make the equation '$y = x + 1$' ***true***.
How can '$y = x + 1$' be true? For a given x-value, the y-value must ***equal*** $x + 1$.
For each x-value, there is ***exactly one*** corresponding y-value—whatever x is, plus 1.
The line is the picture of all the points $(x, x + 1)$, as x varies over all real numbers:

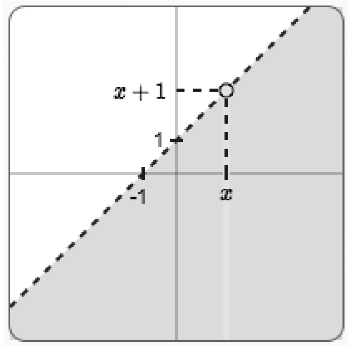


graph of $y = x + 1$: all points of the form $(x, x + 1)$

$$\text{the line '}y = x + 1\text{' is all points of the form:} \quad (\ x\ ,\ \overbrace{x+1}^{\text{the } y\text{-value EQUALS } x+1}\ )$$

Question:
What happens if the verb in the sentence '$y = x + 1$' is changed from '$=$' to $<$, $>$, $\le$, or $\ge$?

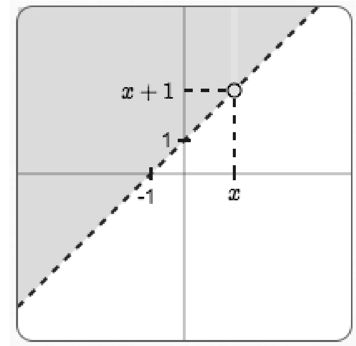
Answer:
You go from a linear ***equation*** in two variables, to a linear ***inequality*** in two variables.
The solution set changes dramatically!
What ***was*** a line now becomes an entire ***half-plane***:



graph of $y < x + 1$:

all points of the form (x, y)
where the y-value
is ***less than*** $x + 1$

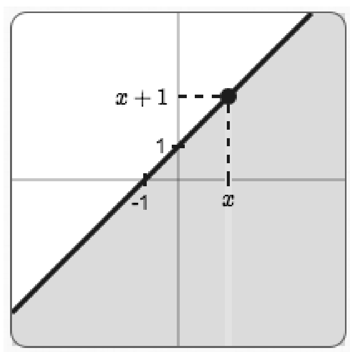
line is dashed;
shade ***below*** the line



graph of $y > x + 1$:

all points of the form (x, y)
where the y-value
is ***greater than*** $x + 1$

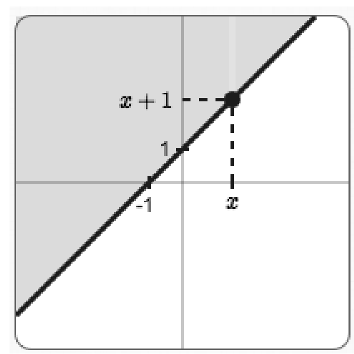
line is dashed;
shade ***above*** the line



graph of $y \le x + 1$:

all points of the form (x, y)
where the y-value
is ***less than or equal to*** $x + 1$

line is solid;
also shade ***below*** the line



graph of $y \ge x + 1$:

all points of the form (x, y)
where the y-value
is ***greater than or equal to*** $x + 1$

line is solid;
also shade ***above*** the line

Important Concepts for Graphing Linear Inequalities in Two Variables

- **DEFINITION: LINEAR INEQUALITY IN TWO VARIABLES**
  A ***linear inequality in two variables*** is a sentence of the form

  $$ax + by + c < 0\,,$$

  where a and b are not ***both*** zero; c can be any real number.
  The inequality symbol can be any of these: $<$, $>$, $\le$, or $\ge$
  Remember: a 'sentence of the form ...' ***really means*** a 'sentence that ***can be*** put in the form ...'

- **EXAMPLES OF LINEAR INEQUALITIES IN TWO VARIABLES**

$3x - 4y + 5 > 0$	$y \le 5x - 1$	$x \ge 2$ (shorthand for $x + 0y \ge 2$)	$y < 5$ (shorthand for $0x + y < 5$)

Key ideas for recognizing linear inequalities in two variables:
  - the verb must be an ***inequality*** symbol: $<$, $\le$, $>$, or $\ge$
  - the variables must be raised ***only to the first power*** :
    no squares, no variables in denominators, no variables under square roots, and so on
  - you don't need to have ***both*** x and y , but you must have at least one of these variables

- **LINEAR INEQUALITIES GRAPH AS HALF-PLANES**

Every linear inequality in two variables graphs as a half-plane:
  - if the verb is $<$ or $>$, the boundary line is ***not included*** (dashed)
  - if the verb is $\le$ or $\ge$, the boundary line ***is included*** (solid)

- **WHICH HALF-PLANE TO SHADE?**

If the linear inequality is in slope-intercept form (like $y < mx + b$), then it's easy to know which half of the line to shade:
  - if the sentence is $y < mx + b$ or $y \le mx + b$, shade BELOW the line
  - if the sentence is $y > mx + b$ or $y \ge mx + b$, shade ABOVE the line

This only works if the inequality is in slope-intercept form!
Of course, you can always ***put*** a sentence in slope-intercept form, by solving for y .
Then, you can use this method.
But, the ‘test point method’ (below) is usually quicker-and-easier, if the sentence isn't already in slope-intercept form.

The Test Point Method for Graphing Linear Inequalities in Two Variables

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So, what about graphing something like $2x - y < 3$, which isn't in slope-intercept form?
You can (if desired) solve for y, and then use the method above:
$$\begin{aligned} 2x - y &< 3 \\ -y &< -2x + 3 \\ y &> 2x - 3 \end{aligned}$$
(Remember to change the direction of the inequality symbol when you multiply/divide by a negative number.)

The graph of $2x - y < 3$ is the same as the graph of $y > 2x - 3$.
Graph the line $y = 2x - 3$ (dashed), and then shade everything above (see right).

However, there's an easier way. Keep reading!
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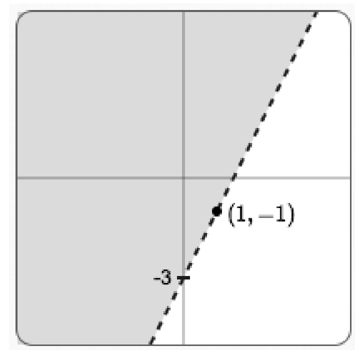

graph of $2x - y < 3$
(which is equivalent to $y > 2x - 3$)
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The 'Test Point Method' is so-called because it involves choosing a 'test point' to decide which side of the line to shade.
The process is illustrated with an example: graphing $2x - y < 3$.
The Test Point Method is usually easiest to use with sentences that aren't in slope-intercept form.

GRAPH, USING THE TEST POINT METHOD: $2x - y < 3$

<table>
<tr>
<td>
Step 1: IDENTIFICATION
Recognize that '$2x - y < 3$' is a linear inequality in two variables.
Therefore, you know the graph is a half-plane.
You need the boundary line; you need to know which side to shade.
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Step 2: BOUNDARY LINE
Graph the boundary line $2x - y = 3$ using the <u>intercept method</u>.
When $x = 0$, $y = -3$.
When $y = 0$, $x = \frac{3}{2}$.
Since the verb in '$2x - y < 3$' is '$<$', this line is not included in the solution set.
Therefore, the line is dashed.
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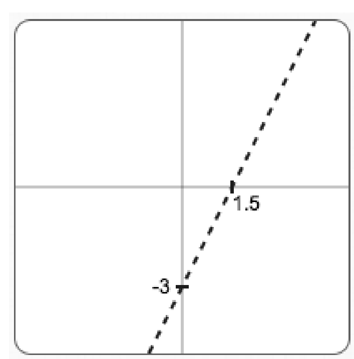

graph the boundary line
using the intercept method
</td>
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Step 3: TEST POINT TO DECIDE WHICH SIDE TO SHADE

Choose a simple point that is NOT on the line.
Whenever $(0,0)$ is available, choose it! Zeroes are ***very*** easy to work with.

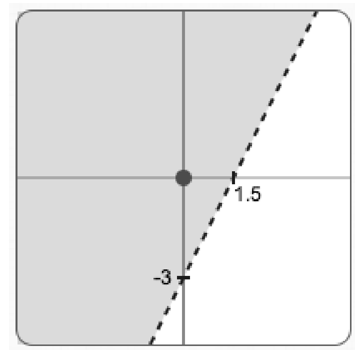
Is $(0,0)$ in the solution set?
Substitute $x=0$ and $y=0$ into the original sentence ($2x-y<3$), to see if it is true or false.
Put a question mark over the inequality symbol, since you're asking a question:

$$2(0)-0\overset{?}{<}3$$

If the result is TRUE, shade the side containing the test point.
If the result is FALSE, shade the other side.

Since ‘ $0<3$ ’ is TRUE, shade the side containing $(0,0)$.
Done!

With so many ***zeroes*** involved in this method, computations can often be done in your head, making this QUICK and EASY!



choose test point $(0,0)$:
since
‘$2(0)-0<3$’ is TRUE,
shade the side containing the test point

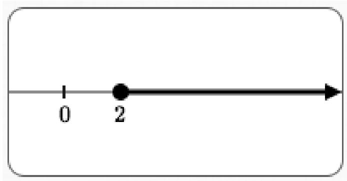
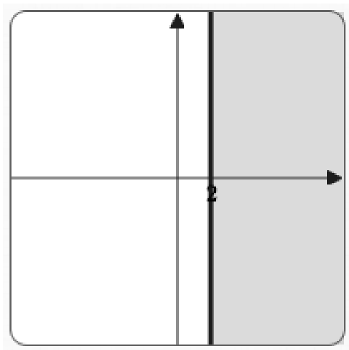
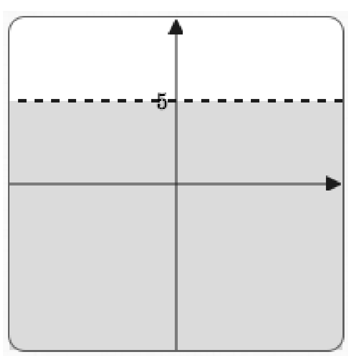
Special Linear Inequalities in Two Variables: You only *see* one variable

Out of context, sentences like ‘ $x\ge 2$ ’ and ‘ $y<5$ ’ can be confusing.
You only ***see*** one variable, but that doesn't necessarily mean that there isn't another variable with a zero coefficient!

Out of context, here are ***clues*** to what is likely wanted:

- ‘Solve: $x\ge 2$ ’ probably wants you to treat this as an inequality in ONE variable.
- ‘Graph: $x\ge 2$ ’ probably wants you to treat this as an inequality in TWO variables.

As shown below, there is a BIG difference in the nature of the solution set!

 graph of $x \geq 2$, viewed as an inequality in ONE variable	Viewed as an inequality in ***one*** variable, the solution set of ‘ $x \geq 2$ ’ is the set of all ***numbers*** that are greater than or equal to 2. The solution set is the interval $[2, \infty)$, shown at left.
 graph of $x \geq 2$, viewed as an inequality in TWO variables	Viewed as an inequality in ***two*** variables, ‘ $x \geq 2$ ’ is really a shorthand for ‘ $x + 0y \geq 2$ ’. The solution set is the set of all ***points*** (x, y), where the x-value is greater than or equal to 2. The y-value can be anything! Here are examples of substitution into ‘ $x + 0y \geq 2$ ’: The point $(2, 5)$ is in the solution set, since ‘ $2 + 0(5) \geq 2$ ’ is TRUE. The point $(3.5, -7.4)$ is in the solution set, since ‘ $3.5 + 0(-7.4) \geq 2$ ’ is TRUE. The graph is the half-plane shown at left. This is the picture of all the points with x-value greater than or equal to 2.
 graph of $y < 5$, viewed as an inequality in TWO variables	Viewed as an inequality in ***two*** variables, ‘ $y < 5$ ’ is really a shorthand for ‘ $0x + y < 5$ ’. The solution set is the set of all ***points*** (x, y), where the y-value is less than 5. The x-value can be anything! Here are examples of substitution into ‘ $0x + y < 5$ ’: The point $(2, 4)$ is in the solution set, since ‘ $0(2) + 4 < 5$ ’ is TRUE. The point $(-7.4, -3)$ is in the solution set, since ‘ $0(-7.4) - 3 < 5$ ’ is TRUE. The graph is the half-plane shown at left. This is the picture of all the points with y-value less than 5.