

### 37. FINISHING UP ABSOLUTE VALUE INEQUALITIES

*solving inequalities  
involving  
absolute value*

This section should feel remarkably similar to the previous two. The only type of absolute value inequality that remains to be investigated is the one involving ‘greater than,’ like these:

$$\begin{aligned} |x| &> 5 \\ |x| &\geq 3 \\ |2 - 3x| &\geq 7 \end{aligned}$$

Each of these inequalities has only a single set of absolute value symbols which is by itself on the left-hand side of the sentence, and has a *variable* inside the absolute value. The verb is either ‘>’ or ‘≥’. As in the previous section, solving sentences like these is easy, if you remember the critical fact that

**absolute value gives distance from 0.**

Keep this in mind as you read the following theorem:

**THEOREM**

*tool for solving  
absolute value  
inequalities  
involving ‘greater than’*

Let  $x \in \mathbb{R}$ , and let  $k > 0$ . Then,

$$\begin{aligned} |x| > k &\iff (x > k \text{ or } x < -k) \\ |x| \geq k &\iff (x \geq k \text{ or } x \leq -k) \end{aligned}$$

*$|x| > k$  is  
an entire class  
of sentences*

Recall first that normal mathematical conventions dictate that ‘ $|x| > k$ ’ represents an entire class of sentences, including  $|x| > 2$ ,  $|x| > 5.7$ , and  $|x| > \frac{1}{3}$ . The variable  $k$  changes from sentence to sentence, but is constant within a given sentence.

**EXERCISES**

1. a. Give three sentences of the form ‘ $|x| > k$ ’ where  $k > 0$ . Use examples different from those given above.
- b. Give three sentences of the form ‘ $(x > k \text{ or } x < -k)$ ’ where  $k > 0$ .

*reminder:  
the mathematical word  
'or'*

Recall that an 'or' sentence is true when at least one of the subsentences is true. That is, 'A or B' is true when A is true, or when B is true, or when both A and B are true.

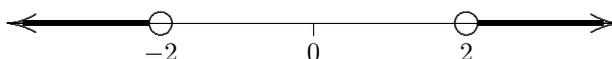
With this in mind, consider the values of  $x$  for which the compound sentence

$$\overbrace{x > 2}^A \text{ or } \overbrace{x < -2}^B$$

is true:

- The sentence is true for any value of  $x$  that makes A true; i.e., for values of  $x$  greater than 2.
- The sentence is true for any value of  $x$  that makes B true; i.e., for values of  $x$  less than  $-2$ .
- There are no values of  $x$  for which A and B are both true at the same time.

Thus, the numbers that make '  $x > 2$  or  $x < -2$  ' true are shaded below:



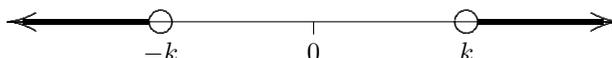
*translating the theorem:  
thought process for  
solving sentences like  
 $|x| > k$*

When you see a sentence of the form ' $|x| > k$ ', here's what you should do:

- Check that  $k$  is a positive number.
- The symbol  $|x|$  represents the distance between  $x$  and 0.
- Thus, you want the numbers  $x$ , whose distance from 0 is greater than  $k$ .

want numbers  $x$ , whose distance from 0  $\overbrace{|x|}$  is greater than  $\overbrace{>}$   $\overbrace{k}$

- You can walk from 0 in two directions: more than  $k$  units to the right, or more than  $k$  units to the left. So, you want all the numbers to the right of  $k$ , together with all the numbers to the left of  $-k$ .



- Thus,  $|x| > k$  is equivalent to  $x > k$  or  $x < -k$ .

*filling in some blanks  
to help your  
thought process*

When you see a sentence like ' $|x| > 7$ ', your thought process should be like filling in the following blanks:

We want the numbers \_\_\_\_\_, whose \_\_\_\_\_ from \_\_\_\_\_ is greater than \_\_\_\_\_. Thus, we want \_\_\_\_\_ to be greater than \_\_\_\_\_ or less than \_\_\_\_\_.

The correctly-filled-in blanks are:

We want the numbers  $x$ , whose distance from 0 is greater than 7. Thus, we want  $x$  to be greater than 7 or less than -7.

**EXERCISES**

2. Fill in the blanks:
  - a. When you look at the sentence ' $|x| > 5$ ', you should think: We want the numbers \_\_\_\_\_, whose \_\_\_\_\_ from \_\_\_\_\_ is greater than \_\_\_\_\_. Thus, we want \_\_\_\_\_ to be greater than \_\_\_\_\_ or less than \_\_\_\_\_.
  - b. When you look at the sentence ' $|z| > \frac{1}{5}$ ', you should think: We want the numbers \_\_\_\_\_, whose \_\_\_\_\_ from \_\_\_\_\_ is greater than \_\_\_\_\_. Thus, we want \_\_\_\_\_ to be greater than \_\_\_\_\_ or less than \_\_\_\_\_.
  - c. When you look at the sentence ' $|x| > k$ ' (with  $k > 0$ ), you should think: We want the numbers \_\_\_\_\_, whose \_\_\_\_\_ from \_\_\_\_\_ is greater than \_\_\_\_\_. Thus, we want \_\_\_\_\_ to be greater than \_\_\_\_\_ or less than \_\_\_\_\_.
3. Give a sentence, not using absolute value symbols, that is equivalent to:
  - a.  $|x| > 3$
  - b.  $|t| \geq 4.2$
4. Give a sentence, using absolute value symbols, that is equivalent to:
  - a.  $x > 7$  or  $x < -7$
  - b.  $t \geq \frac{1}{3}$  or  $t \leq -\frac{1}{3}$
5. Give a precise mathematical statement of the tool that says that a sentence like ' $|x| > 5$ ' can be transformed to the equivalent sentence ' $x > 5$  or  $x < -5$ '.
6. Is the sentence ' $|x| > -6$ ' of the form described in the previous theorem? Why or why not?
7. Can the sentence ' $|x| - 5 > 7$ ' be transformed to a sentence of the form described in the previous theorem? If so, what is the equivalent sentence?

**EXAMPLE**

*solving a sentence  
of the form  
 $|x| > k$*

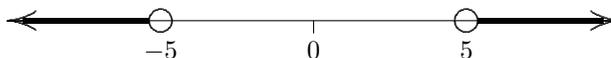
**Example:** Solve:  $|x| > 5$

**Solution:**

$$|x| > 5$$

$$x > 5 \text{ or } x < -5$$

The solution set is shaded below:



solving  
more complicated  
sentences  
of the form  
 $|x| > k$ ;  
 $x$  can be  
ANYTHING!

The power of the tool

$$|x| > k \iff x > k \text{ or } x < -k$$

goes way beyond solving simple sentences like ' $|x| > 5$ '. Since  $x$  can be *any* real number, you should think of  $x$  as merely representing *the stuff inside the absolute value symbols*. Thus, you could think of rewriting the tool as:

$$|\text{stuff}| > k \iff \text{stuff} > k \text{ or } \text{stuff} < -k$$

Thus, we have all the following equivalences:

$$\begin{aligned} \overbrace{|2-3x|}^{\text{stuff}} > 7 &\iff \overbrace{2-3x}^{\text{stuff}} > 7 \text{ or } \overbrace{2-3x}^{\text{stuff}} < -7 \\ \overbrace{|5x-1|}^{\text{stuff}} > 8 &\iff \overbrace{5x-1}^{\text{stuff}} > 8 \text{ or } \overbrace{5x-1}^{\text{stuff}} < -8 \\ \overbrace{|x^2-3x+4|}^{\text{stuff}} > \frac{1}{5} &\iff \overbrace{x^2-3x+4}^{\text{stuff}} > \frac{1}{5} \text{ or } \overbrace{x^2-3x+4}^{\text{stuff}} < -\frac{1}{5} \end{aligned}$$

and so on!

**EXERCISES**

8. For each of the following, write an equivalent sentence that does not use absolute value symbols. Do *not* solve the resulting sentences.
- $|1 + 2x| > 3$
  - $|7x - \frac{1}{2}| \geq 5$
  - $|x^2 - 8| > 0.4$

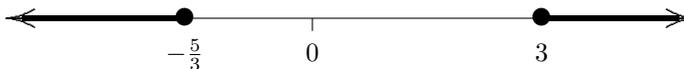
**EXAMPLE**

**Example:** Solve:  $|2 - 3x| \geq 7$

**Solution:** Be sure to write a nice, clean list of equivalent sentences.

$ 2 - 3x  \geq 7$	original sentence
$2 - 3x \geq 7 \text{ or } 2 - 3x \leq -7$	$ x  \geq k \iff x \geq k \text{ or } x \leq -k$
$-3x \geq 5 \text{ or } -3x \leq -9$	subtract 2 from both sides
$x \leq -\frac{5}{3} \text{ or } x \geq 3$	divide both sides by $-3$

The solution set is shaded below:



Checking the boundary values:

$$|2 - 3(-\frac{5}{3})| \stackrel{?}{\geq} 7$$

$$|2 + 5| \stackrel{?}{\geq} 7$$

$$7 \geq 7$$

$$|2 - 3(3)| \stackrel{?}{\geq} 7$$

$$|-7| \stackrel{?}{\geq} 7$$

$$7 \geq 7$$

Checking a value of  $x$  for which the sentence should be false; choose  $x = 0$ :

$$|2 - 3(0)| \stackrel{?}{\geq} 7$$

$$2 \geq 7 \text{ is false!}$$

### EXERCISES

9. Solve. Write a nice, clean list of equivalent sentences. Spot-check and/or check boundary values on your solutions.

a.  $|1 + 2x| > 3$

b.  $|7x - 5| \geq 2$

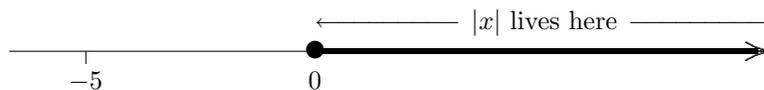
c.  $|4 - 9x| > 1$

What happens if  $k$  is negative in the sentence ' $|x| > k$ '?

What about a sentence like ' $|x| > -5$ ', where the absolute value is greater than a negative number? Notice that this situation is not covered in the previous theorem, since the sentence ' $|x| > k$ ' is only addressed with  $k > 0$ .

Recall that  $|x| \geq 0$  for all real numbers  $x$ . Thus, the sentence ' $|x| > -5$ ' is *always* true! The left-hand side is a nonnegative number, which is guaranteed to be greater than  $-5$ .

$$\underbrace{|x|}_{\geq 0} > \underbrace{-5}_{< 0}$$



Here are a few values of  $x$  substituted into ' $|x| > -5$ ', to illustrate what is happening:

$x$	substitution into ' $ x  > -5$ '	simplifying sentence	true or false?
5	$ 5  > -5$	$5 > -5$	true
-5	$ -5  > -5$	$5 > -5$	true
0	$ 0  > -5$	$0 > -5$	true
2	$ 2  > -5$	$2 > -5$	true

Whenever you see a sentence of the form ' $|x| > a$  negative number', then you should **STOP** and say that the solution set is  $\mathbb{R}$  (the set of all real numbers).

*first step  
when analyzing  
 $|x| > k$  :  
check that  $k > 0$*

Whenever you're working with a sentence of the form ' $|x| > k$ ', you must always check first that  $k > 0$ . If  $k$  is negative, you just stop and say that the sentence is always true. Here are some examples, which illustrate different ways that you can state your answer:

- ' $|x| > -3$ ' is always true.
- ' $|2x - 1| > -5$ ' is true for all real numbers  $x$ .
- ' $|3 - 4x| > -\frac{1}{7}$ ' is true for all  $x \in \mathbb{R}$ .
- ' $|3x - 5x^2 + 7| > -0.4$ ' has solution set  $\mathbb{R}$ .

**EXERCISES**

10. Decide which of the following sentences are always true. If it is sometimes true/sometimes false, so state.
- a.  $|x| > -9$
  - b.  $|x| \geq 0$
  - c.  $|3x - 5| > -4.7$
  - d.  $|1 - 4x| + 5 \geq 0$
  - e.  $-2|x^2 + 3x - 1| < 8$
  - f.  $|9x + 1| - 5 \geq -3$

**EXAMPLE**

*putting a  
sentence in  
standard form  
first*

Sometimes you need a few transformations to get an equivalent sentence in the form  $|x| > k$ , as the next example illustrates. Remember that your goal is always to *isolate* the absolute value; i.e., get it all by itself on one side of the sentence (usually the left-hand side).

Solve:  $5 - 2|3 - 4x| \leq -7$

$5 - 2 3 - 4x  \leq -7$	original sentence
$-2 3 - 4x  \leq -12$	subtract 5 from both sides
$ 3 - 4x  \geq 6$	divide both sides by $-2$ ; change verb
$3 - 4x \geq 6$ or $3 - 4x \leq -6$	$ x  \geq k \iff (x \geq k \text{ or } x \leq -k)$
$-4x \geq 3$ or $-4x \leq -9$	subtract 3
$x \leq -\frac{3}{4}$ or $x \geq \frac{9}{4}$	divide by $-4$ ; change verbs

A spot-check is left to you.

**EXERCISES**

11. Solve and check each of the following sentences. Be sure to write a nice, clean list of equivalent sentences.
- a.  $7 - 5|1 - 2x| < -3$
  - b.  $-3|2x - 1| - 5 \leq -4$
  - c.  $2|3x - 5| - 1 \geq 7$

**EXERCISES**

*web practice*

Go to my homepage <http://onemathematicalcat.org> and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

## SOLUTION TO EXERCISES: FINISHING UP ABSOLUTE VALUE INEQUALITIES

1. a.  $|x| > 1$ ,  $|x| > 7.2$ , and  $|x| > \frac{1}{2}$   
 b.  $x > 1$  or  $x < -1$ ,  $x > 7.2$  or  $x < -7.2$ , and  $x > \frac{1}{2}$  or  $x < -\frac{1}{2}$

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2. a. We want the numbers  $\underline{x}$ , whose distance from  $\underline{0}$  is greater than  $\underline{5}$ . Thus, we want  $\underline{x}$  to be greater than  $\underline{5}$  or less than  $\underline{-5}$ .  
 b. We want the numbers  $\underline{z}$ , whose distance from  $\underline{0}$  is greater than  $\frac{1}{5}$ . Thus, we want  $\underline{z}$  to be greater than  $\frac{1}{5}$  or less than  $\underline{-\frac{1}{5}}$ .  
 c. We want the numbers  $\underline{x}$ , whose distance from  $\underline{0}$  is greater than  $\underline{k}$ . Thus, we want  $\underline{x}$  to be greater than  $\underline{k}$  or less than  $\underline{-k}$ .

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3. a.  $|x| > 3$  is equivalent to  $x > 3$  or  $x < -3$   
 b.  $|t| \geq 4.2$  is equivalent to  $x \geq 4.2$  or  $x \leq -4.2$
4. a.  $x > 7$  or  $x < -7$  is equivalent to  $|x| > 7$   
 b.  $t \geq \frac{1}{3}$  or  $t \leq -\frac{1}{3}$  is equivalent to  $|t| \geq \frac{1}{3}$
5. For all real numbers  $x$ , and for  $k > 0$ ,  $|x| > k$  is equivalent to  $x > k$  or  $x < -k$ .
6. The sentence ' $|x| > -6$ ' is not of the form in the theorem, because  $-6$  is a negative number.
7. The sentence ' $|x| - 5 > 7$ ' can be transformed to ' $|x| > 12$ ' by adding 5 to both sides.

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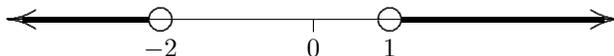
8. a.  $|1 + 2x| > 3$  is equivalent to  $1 + 2x > 3$  or  $1 + 2x < -3$   
 b.  $|7x - \frac{1}{2}| \geq 5$  is equivalent to  $7x - \frac{1}{2} \geq 5$  or  $7x - \frac{1}{2} \leq -5$   
 c.  $|x^2 - 8| > 0.4$  is equivalent to  $x^2 - 8 > 0.4$  or  $x^2 - 8 < -0.4$

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9. a.

$$\begin{aligned}
 &|1 + 2x| > 3 \\
 &1 + 2x > 3 \quad \text{or} \quad 1 + 2x < -3 \\
 &2x > 2 \quad \text{or} \quad 2x < -4 \\
 &x > 1 \quad \text{or} \quad x < -2
 \end{aligned}$$

The solution set is:



Spot-check:

Choose a value of  $x$  greater than 1; choose (say)  $x = 2$ :

$$\begin{aligned}
 &|1 + 2(2)| \stackrel{?}{>} 3 \\
 &5 > 3 \quad \text{is true}
 \end{aligned}$$

Choose a value of  $x$  between  $-2$  and  $1$ ; choose (say)  $x = 0$ :

$$\begin{aligned}
 &|1 + 2(0)| \stackrel{?}{>} 3 \\
 &1 > 3 \quad \text{is false}
 \end{aligned}$$

Choose a value of  $x$  less than  $-2$ ; choose (say)  $x = -3$ :

$$|1 + 2(-3)| \stackrel{?}{>} 3$$

$$5 > 3 \text{ is true}$$

b.

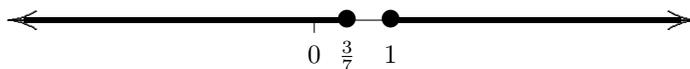
$$|7x - 5| \geq 2$$

$$7x - 5 \geq 2 \text{ or } 7x - 5 \leq -2$$

$$7x \geq 7 \text{ or } 7x \leq 3$$

$$x \geq 1 \text{ or } x \leq \frac{3}{7}$$

The solution set is:



Spot-check:

Choose a value of  $x$  greater than 1; choose (say)  $x = 2$ :

$$|7(2) - 5| \stackrel{?}{\geq} 2$$

$$9 \geq 2 \text{ is true}$$

Choose a value of  $x$  between  $\frac{3}{7}$  and 1; choose (say)  $x = \frac{1}{2}$ :

$$|7(\frac{1}{2}) - 5| \stackrel{?}{\geq} 2$$

$$1.5 \geq 2 \text{ is false}$$

Choose a value of  $x$  less than  $\frac{3}{7}$ ; choose (say)  $x = 0$ :

$$|7(0) - 5| \stackrel{?}{\geq} 2$$

$$5 \geq 2 \text{ is true}$$

c.

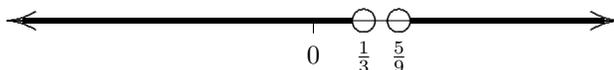
$$|4 - 9x| > 1$$

$$4 - 9x > 1 \text{ or } 4 - 9x < -1$$

$$-9x > -3 \text{ or } -9x < -5$$

$$x < \frac{1}{3} \text{ or } x > \frac{5}{9}$$

The solution set is:



Spot-check:

Choose a value of  $x$  greater than  $\frac{5}{9}$ ; choose (say)  $x = 1$ :

$$|4 - 9(1)| \stackrel{?}{>} 1$$

$$5 > 1 \text{ is true}$$

Choose a value of  $x$  between  $\frac{1}{3} = \frac{3}{9}$  and  $\frac{5}{9}$ ; choose (say)  $x = \frac{4}{9}$ :

$$|4 - 9(\frac{4}{9})| \stackrel{?}{>} 1$$

$0 > 1$  is false

Choose a value of  $x$  less than  $\frac{1}{3}$ ; choose (say)  $x = 0$ :

$$|4 - 9(0)| \stackrel{?}{>} 1$$

$4 > 1$  is true

10. (a)  $|x| > -9$  is always true.

(b)  $|x| \geq 0$  is true for all real numbers  $x$ .

(c)  $|3x - 5| > -4.7$  is true for all  $x \in \mathbb{R}$ .

(d)  $|1 - 4x| + 5 \geq 0$  is equivalent to  $|1 - 4x| \geq -5$ , which has solution set  $\mathbb{R}$ .

(e)  $-2|x^2 + 3x - 1| < 8$  is equivalent to  $|x^2 + 3x - 1| > -4$ , which is always true.

(f)  $|9x + 1| - 5 \geq -3$  is equivalent to  $|9x + 1| \geq 2$ ; it is sometimes true, sometimes false.

11. a.

$$7 - 5|1 - 2x| < -3$$

$$-5|1 - 2x| < -10$$

$$|1 - 2x| > 2$$

$$1 - 2x > 2 \quad \text{or} \quad 1 - 2x < -2$$

$$-2x > 1 \quad \text{or} \quad -2x < -3$$

$$x < -\frac{1}{2} \quad \text{or} \quad x > \frac{3}{2}$$

Spot-checks: (A compact way of writing down the spot-checks is illustrated here.)

$x = -1$  (should be true):  $7 - 5|1 - 2(-1)| = 7 - 5|1 + 2| = 7 - 5(3) = 7 - 15 = -8$  is less than  $-3$

$x = 0$  (should be false):  $7 - 5|1 - 2(0)| = 7 - 5|1| = 7 - 5 = 2$  is not less than  $-3$

$x = 2$  (should be true):  $7 - 5|1 - 2(2)| = 7 - 5|-3| = 7 - 5(3) = 7 - 15 = -8$  is less than  $-3$

b.

$$-3|2x - 1| - 5 \leq -4$$

$$-3|2x - 1| \leq 1$$

$$|2x - 1| \geq -\frac{1}{3}$$

always true!

c.

$$2|3x - 5| - 1 \geq 7$$

$$2|3x - 5| \geq 8$$

$$|3x - 5| \geq 4$$

$$3x - 5 \geq 4 \quad \text{or} \quad 3x - 5 \leq -4$$

$$3x \geq 9 \quad \text{or} \quad 3x \leq 1$$

$$x \geq 3 \quad \text{or} \quad x \leq \frac{1}{3}$$

Spot-checks:

$x = 0$  (should be true):  $2|3(0) - 5| - 1 = 2|-5| - 1 = 2(5) - 1 = 10 - 1 = 9$  is greater than or equal to 7.

$x = 1$  (should be false):  $2|3(1) - 5| - 1 = 2|-2| - 1 = 2(2) - 1 = 4 - 1 = 3$  is not greater than or equal to 7.

$x = 4$  (should be true):  $2|3(4) - 5| - 1 = 2|7| - 1 = 2(7) - 1 = 14 - 1 = 13$  is greater than or equal to 7.