30. TRANSFORMING TOOL #1 (the Addition Property of Equality)

sentences that look different, but always have the same truth values

In an earlier section, we saw that the sentences '$2x - 3 = 0$' and '$x = \frac{3}{2}$' certainly *look* different, but are 'the same' in a very important way: they *always* have the same truth. No matter what number is chosen for x, the sentences are true at the same time, and false at the same time.

What can you DO to a sentence that will make it LOOK different, but not change its truth values?

Here's the question to be explored in this section and the next:

How do we get *FROM* the harder equation '$2x - 3 = 0$'
TO the simpler equation '$x = \frac{3}{2}$'?

That is, what can you *do* to an equation that will make it *look* different, but not change its truth?

Two transforming 'tools' are needed to change '$2x - 3 = 0$' into '$x = \frac{3}{2}$'. One of these tools (the *addition property of equality*) is discussed in this section; the other (the *multiplication property of equality*) is discussed in the following section.

Here's the way you would be told about one of the most commonly-used transforming tools for equations, using the language of mathematics:

THEOREM

Addition Property of Equality

For all real numbers a, b, and c,

$$a = b \iff a + c = b + c\ .$$

EXERCISES

1. What is a 'theorem'?

2. What are the universal sets for a, b, and c in the previous theorem? How do you know?

3. How would you read aloud the displayed sentence '$a = b \iff a+c = b+c$'? (In particular, how do you read the symbol '$\iff$'?)

What is this saying?

FACTS versus COMMANDS

To a person not trained in reading mathematics, the information contained in this theorem is completely inaccessible. If you don't understand the language in which an idea is being expressed, then you can't use the idea.

In mathematics, you are told how to *do* things by being given FACTS (declarative mode). However, most beginning students of mathematics like to be told how to *do* things by being given COMMANDS ('Do this ... Do this ...'; imperative mode). This incompatibility in the way that information is given in mathematics, and the way that students like to receive information, is a major source of math anxiety.

Warren Esty introduced the declarative/imperative mode distinction in his excellent book **The Language of Mathematics**. Information about this book is available at the web page

http://www.math.montana.edu/~umsfwest/

two steps to take upon reading any theorem

When you first read *any* theorem, there are two questions that you must ask:

- What is the theorem telling you that you can *DO*?
  (That is, take the *fact*, and translate it into what you can *do*. We'll call this 'translating the theorem'.)
- *WHY* is the theorem true?
  (Don't just blindly accept that something is true. Find out *why* it is true.)

'for all' sentences; (generalizations)

Before translating the Addition Property of Equality, it will help to have some better understanding of 'for all' sentences.

The overall structure of the previous theorem is:

> For all *these values of the variable(s)* ,
> (*some sentence involving the variable(s)*).

A mathematical sentence that begins with the words 'for all ... ' is formally known as a **generalization** (since a 'general' statement is being made), but is informally called a 'for all' sentence.

'for all' sentences can only be TRUE or FALSE

Unlike other mathematical sentences involving variables, a 'for all' sentence *cannot be sometimes true/sometimes false*. It can only be *true* or *false*. Next, we discuss when it is true; and when it is false.

When is a 'for all' sentence TRUE?

When is a 'for all' sentence *true*?

Roughly, a 'for all' sentence is *true* when you can append the two words 'IS TRUE' on the end of the sentence, and get a true statement as a result:

> For all $\overbrace{\textit{these values of the variable(s)}}^{(*)}$,
> $\overbrace{\textit{some sentence involving the variable(s)}}^{(**)}$ **IS TRUE .**

More precisely: if a 'for all' sentence is true, then the following condition holds; and if the following condition holds, then a 'for all' sentence is true:

> **no matter what choice of variable(s) is made from (*), substitution of these variable(s) into (**) yields a true sentence.**

When is a 'for all' sentence FALSE?

When is a 'for all' sentence *false*?

A 'for all' sentence is *false* when there is some choice of variable(s) from (*) for which (**) is false.

More precisely: if a 'for all' sentence is false, then the following condition holds; and if the following condition holds, then a 'for all' sentence is false:

> **There is at least one choice of variable(s) from (*) that makes (**) false.**

EXERCISE

4. What does it mean for there to be '*at least one* choice of variable(s) ... '? That is, translate the words 'at least one'.

The following examples explore two simple 'for all' sentences: one true, and one false.

EXAMPLE

a TRUE 'for all' sentence

This 'for all' sentence is true:

For all real numbers x and y , $x + y = y + x$.

Why is it true? When you append the two words 'IS TRUE' to the end of the sentence, as follows, a true statement results:

For all real numbers x and y , *the sentence* $x + y = y + x$ *IS TRUE*.

That is, no matter what real numbers are chosen for x and y , substitution of these numbers into '$x + y = y + x$' yields a true sentence.

Here, you are being informed that you can 'commute' the numbers in any addition problem, without changing the outcome. This fact is formally referred to as the **Commutative Property of Addition**.

EXAMPLE

a FALSE 'for all' sentence

This 'for all' sentence is false:

$$\text{For all } \overbrace{\text{real numbers } x}^{(*)},$$

$$\overbrace{\text{the distance between } x \text{ and zero on a number line is equal to } x}^{(**)}.$$

Why is it false? Because there is at least one choice from (*) that makes (**) false.

For example, choose x to be -2. Then, the sentence

'the distance between -2 and zero on a number line is equal to -2'

is false. (In reality, the distance between -2 and zero is 2.)

Or, choose, x to be -5.3. Then, the sentence

'the distance between -5.3 and zero on a number line is equal to -5.3'

is again false. (In reality, the distance between -5.3 and zero is 5.3.)

BE CAREFUL!

BE CAREFUL! When a 'for all' sentence is false, you *cannot conclude* that (**) is *always* false! Indeed, there may be many choices from (*) for which (**) is true. It only means that *there exists a choice* from (*) for which (**) is false.

In this example, a modification of the universal set for x results in a *true* 'for all' sentence:

For all $x \ge 0$, the distance between x and zero on a number line is equal to x.

EXERCISES

5. Decide whether each 'for all' sentence is TRUE or FALSE.

If the 'for all' sentence is false, produce a choice of variable(s) which proves that it is false.

(a) For all real numbers x, y, and z, $x + (y + z) = (x + y) + z$.

(b) For all real numbers x and y, $x - y = y - x$.

(c) For all real numbers x and y, $xy = yx$.

(d) For all nonzero real numbers x and y, $\frac{x}{y} = \frac{y}{x}$.

translating the theorem

Now, let's translate the Addition Property of Equality. Afterwards, we'll discuss why it is true. The theorem is repeated below for your convenience:

THEOREM

Addition Property of Equality

For all real numbers a, b, and c,

$$a = b \iff a + c = b + c .$$

a first-level translation

A first-level translation might go something like this:

'No matter what real numbers are currently being held by a, b, and c, the compound sentence '$a = b \iff a + c = b + c$' is true.'

Next, we must ask:

What does it mean for the sentence '$a = b \iff a + c = b + c$' to be true?

Answer: It means that the subsentences '$a = b$' and '$a + c = b + c$' have precisely the same truth values. For *every* choice of real numbers a, b, and c:

If '$a = b$' is true, so is '$a + c = b + c$'.

If '$a = b$' is false, so is '$a + c = b + c$'.

If '$a + c = b + c$' is true, so is '$a = b$'.

If '$a + c = b + c$' is false, so is '$a = b$'.

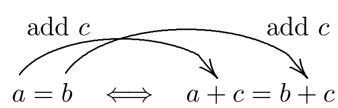
OKAY: even though '$a = b$' and '$a + c = b + c$' may *look* different, they have the same truth values. So what?

the critical part of the translation

This leads us to the critical part of the translation:

What did you *do* to '$a = b$' to transform it into '$a + c = b + c$'?

ANSWER: You added c to both sides of the equation.



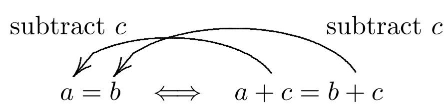
Hence the first part of the translation:

You can add the same number to both sides of an equation, and this won't change the truth of the equation.

Continuing the translation:

What did you *do* to '$a + c = b + c$' to transform it into '$a = b$'?

ANSWER: You subtracted c from both sides of the equation.



Hence the rest of the translation:

You can subtract the same number from both sides of an equation, and this won't change the truth of the equation.

your translation should NOT use 'a', 'b', and 'c'

Notice that the translation of this theorem did NOT mention the variable names: 'a', 'b', and 'c'. The variables are just used, locally, to *encode* information about *what you can DO*. The theorem could equally well be stated using the variables x, y, and z; but it would still be 'translated' in exactly the same way.

the full translation of the Addition Property of Equality

Combining results, here's the way an instructor of mathematics might translate the Addition Property of Equality, to tell students what they can *do*:

You can add (or subtract) the same number to (or from) both sides of an equation, and this won't change the truth of the equation.

Since the *Addition* Property of *Equality* has to do with *adding* numbers to both sides in a statement of *equality*, the name is appropriate.

★
subtraction isn't needed: everything can be done with addition

Actually, subtraction is superfluous: everything can be done with addition. That is, for all real numbers a and b,

$$a - b = a + (-b) \ .$$

To subtract a number is the same as adding its opposite. Thus, a translation of the previous theorem might simply be: 'You can add the same number to both sides of an equation, and this won't change the truth of the equation.' By stating that it works for addition, you're also stating that it works for subtraction.

★
truth **value** *(singular) versus truth* **values** *(plural); two different conceptual levels*

When a mathematician compares a pair of equations, such as

'$a = b$' and '$a + c = b + c$',

they are usually being 'compared' on two different conceptual levels.

For any *particular* values of a, b, and c, the two equations will have the same truth **value** (singular). They're both true, or they're both false.

On a higher conceptual level, no matter *what* values are being held by a, b, and c, the two equations will always have the same truth **values** (plural). For some choices of variables, they're both true. For some choices of variables, they're both false.

To completely avoid the problem of whether to say

> adding the same number to both sides of an equation won't change the truth **value** of the equation ...

or

> adding the same number to both sides of an equation won't change the truth **values** of the equation ...

the author has chosen to 'cover' both conceptual levels by merely saying

> adding the same number to both sides of an equation won't change the **truth** of the equation.

There is a strong geometric reason as to why the Addition Property of Equality is true. Before discussing *why* it is true, however, let's use it to solve a very simple equation. Remember that to *solve* an equation means to determine when the equation is *true*.

EXAMPLE
solving a very simple equation

SOLVE: $x - 2 = 5$

Most people can solve this equation by inspection, because it's so simple. You need only think: 'What number, minus 2, gives 5?' The answer is of course 7.

However, let's solve it by using the Addition Property of Equality. We'll transform the original equation into one that's even *easier* to work with. The lines are numbered so that they can be easily referred to in the ensuing discussion. 'LHS' refers to the **L**eft-**H**and **S**ide of the equation; 'RHS' refers to the **R**ight-**H**and **S**ide of the equation.

line 1: $x - 2 = 5$ (Start with the original equation.)

line 2: $\overbrace{x-2}^{\text{previous LHS}} + 2 = \overbrace{5}^{\text{previous RHS}} + 2$ (Add 2 to both sides.)

line 3: $x = 7$ (Simplify each side.)

The equations in lines 1, 2, and 3 all look different. That is, '$x - 2 = 5$' looks different from '$x - 2 + 2 = 5 + 2$' which looks different from '$x = 7$'. However, no matter what number is chosen for x, they all have the same truth values. The equation in line 3 is simplest. The only time that '$x = 7$' is true is when x is 7. Consequently, the only time that '$x - 2 = 5$' is true is when x is 7.

key ideas used in the previous example:

There are several key ideas used in the previous example (and in a wide variety of similar examples):

transform equation to the form '$x = (\text{some number})$'

- *TRANSFORM THE EQUATION TO THE FORM*

  $$\text{'}x = (\textit{some number})\text{'}.$$

  That is, you want to end up with an equation where the variable is all by itself on one side, with a specific number on the other side.

  It doesn't matter which side you get the variable on: '$x = 2$' gives precisely the same information as '$2 = x$'. However, it is conventional to put the variable on the left-hand side.

undo subtraction with addition

- *UNDO SUBTRACTION WITH ADDITION*. For example, to undo the operation 'subtract 2', you would apply the transformation 'add 2'. Here's how this idea was used in the previous example:
  - The left-hand side started as '$x - 2$', where 2 is subtracted from x.
  - We wanted x all by itself.
  - To undo 'subtract 2', the transformation 'add 2' was applied.

undo addition with subtraction

- *UNDO ADDITION WITH SUBTRACTION*. For example, to undo the operation 'add 5', you would apply the transformation 'subtract 5'.

The next exercise provides practice with these 'undoing' transformations.

EXERCISE

6. What would you *do* to each equation, to transform it to an equation of the form

$$\text{'}(\textit{some variable}) = (\textit{some number})\text{'}?$$

The first one is done for you.

(sample) $x + 2 = 5$

Answer: Subtract 2 from both sides. (To undo 'add 2', apply the transformation 'subtract 2'.)

(a) $x - 4 = 10$
(b) $x + 7 = 4$
(c) $11 = x - 7$
(d) $6 = 7 + x$
(e) $3 + t = 5$
(f) $5 = 4 + t$

acceptable layouts to use when solving an equation

There are a variety of ways that people write down the process of going from the equation '$x - 2 = 5$' to the equation '$x = 7$'. Here are several acceptable ways, with comments on each:

format (1)

$$x - 2 = 5$$
$$x - 2 + 2 = 5 + 2$$
$$x = 7$$

Write a list of equivalent equations. Show the transformation 'add 2 to both sides' as an extra equation in the list. Writing this extra equation takes a bit more space, but can prevent lots of errors.

format (2)

$$\begin{array}{r} x - 2 = 5 \\ +2 \quad +2 \\ \hline x = 7 \end{array}$$

Show the transformation 'add 2 to both sides' by vertical addition. Although some beginning students choose this format, it is discouraged later on; better to choose one of the other formats discussed here.

format (3)

$$\begin{array}{r} x - 2 = 5 \\ x = 7 \end{array} \quad \text{add 2}$$

Here, an arrow is used to illustrate that the transformation 'add 2 to both sides' takes the first equation to the second equation. The actual addition is done mentally, and the result is just written down.

format (4)

$$x - 2 = 5$$
$$x = 7$$

Here, the transformation 'add 2 to both sides' is done mentally; the result is just written down. This is the most concise format.

format (5)

For all real numbers x,

$$x - 2 = 5 \iff x = 7\ .$$

All the other methods are really just shorthands for this. Although this is the most correct way to write the solution process, it is also the longest, and it doesn't exhibit what you *did* in going from one step to the next.

the author's preferred formats

Formats (1), (3), or (4) will be used for the remainder of the text.

The general format to be used for solving any mathematical sentence is summarized in the following box:

> **SOLVING A SENTENCE**
> **PREFERRED FORMAT**
>
> To *solve a sentence* means to find the choice(s) that make the sentence *true*. This is done by correctly 'transforming' the sentence into an equivalent (or nearly equivalent) one that is easier to work with.
>
> Whenever you *solve a sentence*, you should write a list of equivalent (or nearly equivalent) sentences, ending with one so simple, that it can be solved by inspection.
>
> *original sentence*
> *transformed sentence 1*
> *transformed sentence 2*
> ⋮
> *final sentence*

★ *'nearly equivalent'; extraneous solutions*	By saying 'nearly equivalent', the author is allowing for the fact that certain operations (like multiplying by zero, or squaring) can 'add' a solution (that is, take a false or undefined sentence to a true sentence). Students learn that they must watch for extraneous solutions in certain circumstances.

EXERCISE	7. Solve each equation. Use all five formats; decide on a personal format preference. (a) $x - 1 = 4$ (b) $5 = 3 + x$

WHY is the Addition Property of Equality true?

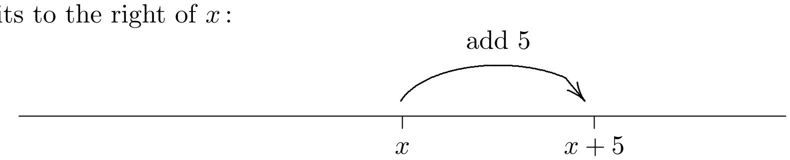
Next, we'll discuss *why* the Addition Property of Equality is true. That is, *why* does adding/subtracting the same number to/from both sides of an equation preserve the truth of the equation?

two key ideas

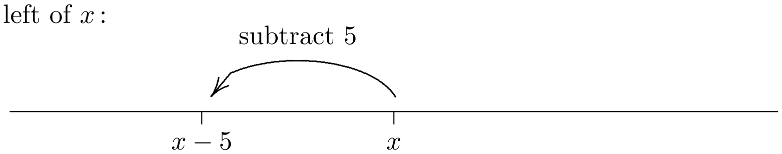
Two key ideas are needed:

- Recall the number-line interpretation of equality (and non-equality) of numbers: if '$a = b$' is true, then a and b live at the same place on a number line; if '$a = b$' is false, then a and b live at different places.
- Let x represent any real number. Then, adding/subtracting a number to/from x results in a movement on the number line.

  For example, adding 5 to x moves it to the new number $x + 5$ which is 5 units to the right of x:



  Subtracting 5 from x moves it to the new number $x - 5$ which is 5 units to the left of x:



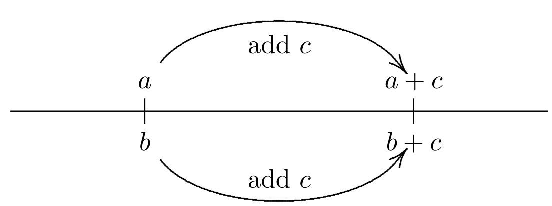
applying the key ideas:

if '$a = b$' is true, then '$a + c = b + c$' is true

Let's apply these key ideas to understanding the Addition Property of Equality. Remember: we're trying to show that the sentences '$a = b$' and '$a + c = b + c$' always have the same truth values.

- Suppose that '$a = b$' is true. Then, a and b live at the same place on a number line.
- Adding c to both numbers moves them both by the same amount.
- Since they started at the same place, they'll end up at the same place. Thus, '$a + c = b + c$' is also true.

The sketch below assumes that c is a positive number:



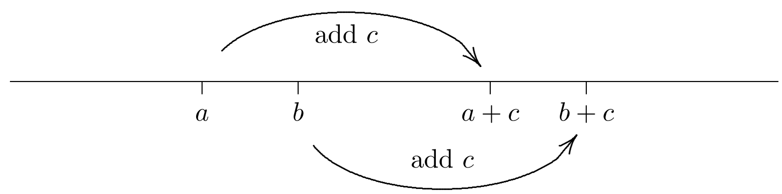
So, whenever '$a = b$' is true, '$a + c = b + c$' is also true. If a and b start at the same place, and are moved by the same amount, then they end up at the same place.

if '$a = b$' is false, then '$a + c = b + c$' is false

One more time:

- Suppose that '$a = b$' is false. Then, a and b live at different places on a number line.
- Adding c to both numbers moves them both by the same amount.
- Since they started at different places, they'll end up at different places. Thus, '$a + c = b + c$' is also false.

The sketch below assumes that c is a positive number:



So, whenever '$a = b$' is false, '$a + c = b + c$' is also false. If a and b start at different places, and are moved by the same amount, then they end up at different places.

EXERCISE

8. Thus far, it has been shown that:
   - if '$a = b$' is true, so is '$a + c = b + c$' .
   - if '$a = b$' is false, so is '$a + c = b + c$' .

   Now, you try these:

   (a) Suppose that '$a + c = b + c$' is true. Argue that '$a = b$' must also be true.

   (b) Suppose that '$a + c = b + c$' is false. Argue that '$a = b$' must also be false.

★ *a shorter proof for experienced mathematicians*

To prove the equivalence

$$a = b \iff a + c = b + c \ ,$$

it is (in this situation) only necessary to prove the implication

$$a = b \ \Rightarrow \ a + c = b + c \ . \qquad (*)$$

For suppose (*) is true. Then,

$$\begin{aligned} a + c = b + c \ &\Rightarrow \ (a + c) + (-c) = b + c + (-c) \\ &\Rightarrow \ a = b \ . \end{aligned}$$

The 'redundant' argument preceding this boxed material should be extra convincing to students. It also allows the author to avoid a discussion of implications.

solving a more complicated equation

Let's solve a slightly more complicated equation using the Addition Property of Equality.

Most equations require more than one 'transforming tool' to arrive at the desired form '$x = $ (*some number*)' . However, the following example is contrived so that it requires *only* the Addition Property of Equality.

EXAMPLE SOLVE: $3x - 1 = 2x + 7$

solving a more complicated equation

SOLUTION (using format (1)):

$3x - 1 = 2x + 7$	Start with the original equation.
$3x - 1 - 2x = 2x + 7 - 2x$	We want x on only one side. To clear the $2x$ on the RHS, undo '$\text{add } 2x$' with '$\text{subtract } 2x$'. Remember that $2x$ is a real number: subtracting *any* real number from both sides of an equation doesn't change the truth of the equation.
$x - 1 = 7$	Simplify both sides: $3x$ (three ex), take away $2x$ (two ex), leaves $1x$ (one ex). Also, '$1x$' goes by the simpler name 'x' (since multiplying by 1 doesn't change a number).
$x - 1 + 1 = 7 + 1$	We want x all by itself. Undo 'subtract 1' with 'add 1'.
$x = 8$	Simplify both sides.

checking the solution

The final equation is true only when x is 8. Therefore, the original equation (and every other equation in the list) is true only when x is 8. Substituting 8 for x in the original equation verifies that it is indeed true:

$$\begin{aligned}\overbrace{3x - 1 \;&=\; 2x + 7}^{\text{original equation}}\\ 3(8) - 1 \;&\stackrel{?}{=}\; 2(8) + 7\\ 24 - 1 \;&\stackrel{?}{=}\; 16 + 7\\ 23 \;&=\; 23\end{aligned}$$

This substitution into the original equation is known as **checking the solution**.

Notice that it would have been extremely difficult to solve the original equation by inspection. You would need to think: what number, times 3, minus 1, is the same as that number, times 2, plus 7?

However, by transforming the original equation to the *equivalent* equation '$x = 8$', we can learn about the truth of the (hard) original equation by instead investigating the truth of the final (simple) equation. We learn two things:

- The number 8 makes the original equation true.
- The number 8 is the *only* number that makes the original equation true.

Everybody makes mistakes, so it's always desirable to check your solution(s) after solving a sentence. The checking format illustrated in the previous example is repeated below, with comments:

★ *another reason for checking solution(s)*

There's another reason to get in the habit of checking your solution(s). When the transforming process involves something that might *add* a solution (like squaring both sides, or multiplying by something that has the potential of equaling zero), then there might be extraneous solutions, which will need to be discarded.

desired format for checking a solution

$3(8) - 1 \overset{?}{=} 2(8) + 7$	Substitute the (hopeful) solution into the original equation. You're *hoping* that the resulting equation is true: but, if you made a mistake in the solution process, then it could very well be false. The question mark (?) over the equal sign covers you, in case a mistake was indeed made. Basically, you're saying: is the left-hand side equal to the right-hand side?
$24 - 1 \overset{?}{=} 16 + 7$	Simplify each side separately. Keep the question mark over the equal sign until you're certain that both sides are equal.
$23 = 23$	Drop the question mark when it is clear that both sides are indeed equal. The check is done!

don't apply transforming tools when checking a solution

Don't apply any 'transforming tools' when checking a solution. That is, after writing, say,

$$3(8) - 1 \overset{?}{=} 2(8) + 7\ ,$$

you would NOT want to add 1 to both sides. Just work with each side separately, using your calculator as needed to do the required arithmetic.

if you did make a mistake...

If the 'check' leads you to a false equation (like '$19 = 23$'), then a mistake was made in the solving or checking process (or both). In this situation, put a slash through the equal sign ('$19 \neq 23$') to make it a true sentence, and then go back and find your mistake.

END-OF-SECTION EXERCISES

For problems 9–12: SOLVE each of the following equations. (Each equation can be solved using only the Addition Property of Equality.) Check your solution. Use acceptable formats for both the solution and the check.

9. $4x = 5 + 3x$
10. $x + 4 = 2x - 1$
11. $3t - 7 + 2t = 4t + 5$
12. $1 = 8 - t$

EXERCISES *web practice*

Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SECTION SUMMARY
TRANSFORMING TOOL #1
(the Addition Property of Equality)

NEW IN THIS SECTION	HOW TO READ	MEANING
the Addition Property of Equality		For all real numbers a, b, and c, $$a = b \iff a + c = b + c\ .$$ Translation: adding/subtracting the same number to/from both sides of an equation doesn't change the truth of the equation.
FACTS (declarative mode) versus COMMANDS (imperative mode)		Mathematicians tell you how to *do* things by stating facts (declarative mode). Many beginning students like to be told how to *do* things by being given commands (imperative mode). This incompatibility in the way that mathematicians give information, and the way that students like to receive information, is a major source of math anxiety.
'for all' sentences; (generalizations)		A 'for all' sentence (formally called a *generalization*) is a sentence of the form: For all $\overbrace{\textit{these values of the variable(s)}}^{(*)}$, $\overbrace{\textit{(some sentence involving the variable(s))}}^{(**)}$. A 'for all' sentence *cannot be sometimes true/sometimes false*. It can only be *true* or *false*. A 'for all' sentence is *true* when the following condition holds: no matter what choice of variable(s) is made from (*), substitution of these variable(s) into (**) yields a true sentence. A 'for all' sentence is *false* when there is *at least one* choice of variable(s) from (*) that makes (**) false.

NEW IN THIS SECTION	HOW TO READ	MEANING
solving a sentence; preferred format		To *solve a sentence* means to find the choice(s) that make the sentence *true*. This is done by correctly 'transforming' the sentence into an equivalent (or nearly equivalent) one that is easier to work with. Whenever you *solve a sentence*, you should write a list of equivalent (or nearly equivalent) sentences, ending with one so simple, that it can be solved by inspection. *original sentence* *transformed sentence 1* *transformed sentence 2* ⋮ *final sentence*
a preferred format for solving a sentence: show the transformation as an extra sentence in the list		EXAMPLE: $x - 2 = 5$ $x - 2 + 2 = 5 + 2$ $x = 7$ Write a list of equivalent equations. Show the transformation 'add 2 to both sides' as an extra equation in the list. Writing this extra equation takes a bit more space, but can prevent lots of errors.
a preferred format for solving a sentence: show the transformation as an arrow		EXAMPLE: $x - 2 = 5$ ⤵ add 2 $x = 7$ Use an arrow to illustrate that the transformation 'add 2 to both sides' takes the first equation to the second equation. The actual addition is done mentally, and the result is just written down.
most concise format for solving a sentence		EXAMPLE: $x - 2 = 5$ $x = 7$ The transformation 'add 2 to both sides' is done mentally; the result is just written down.

NEW IN THIS SECTION	HOW TO READ	MEANING
checking a solution; preferred format		To **check a solution** means to substitute it into the original sentence, and verify that a true sentence results. EXAMPLE: To check that 8 is a solution of the equation ‘ $3x - 1 = 2x + 7$ ’, you would write: $\begin{aligned} 3(8) - 1 &\overset{?}{=} 2(8) + 7 \\ 24 - 1 &\overset{?}{=} 16 + 7 \\ 23 &= 23 \end{aligned}$ The question mark (?) over the equal sign covers you, in case a mistake was indeed made. Basically, you’re saying: is the left-hand side equal to the right-hand side? Drop the question mark when it is clear that both sides are indeed equal. If the ‘check’ leads you to a false equation (like ‘ $19 = 23$ ’), then a mistake was made in the solving or checking process (or both). In this situation, put a slash through the equal sign (‘ $19 \ne 23$ ’) to make it a true sentence, and then go back and find your mistake.

SOLUTIONS TO EXERCISES: TRANSFORMING TOOL #1 (the Addition Property of Equality)

IN-SECTION EXERCISES:

1. A 'theorem' is a mathematical result that is both TRUE and IMPORTANT (★ that has been proved).

2. The universal set for each variable is $\mathbb{R}$ (the set of real numbers). The phrase 'For all *real numbers* ...' gives us this information.

3. 'a equals b (*slight pause*) is equivalent to (*slight pause*) a plus c equals b plus c';
or
'a equals b if and only if a plus c equals b plus c'

4. In this context, 'at least one' means 'one, or more than one'.

5. (a) TRUE. In an addition problem, the grouping of the numbers does not affect the result.

(b) FALSE. For example, choose $x = 1$ and $y = 2$. Then, the sentence '$1 - 2 = 2 - 1$' is false.

(c) TRUE. You can commute the numbers in a multiplication problem, without changing the result. This fact is formally referred to as the **Commutative Property of Multiplication**.

(d) FALSE. For example, choose $x = 1$ and $y = 2$. Then, the sentence '$\frac{1}{2} = \frac{2}{1}$' is false.

6. (a) Add 4 to both sides. (To undo 'subtract 4', apply the transformation 'add 4'.)

(b) Subtract 7 from both sides. (To undo 'add 7', apply the transformation 'subtract 7'.)

(c) Add 7 to both sides. (To undo 'subtract 7', apply the transformation 'add 7'.)

(d) Subtract 7 from both sides. (To undo 'add 7', apply the transformation 'subtract 7'.)

(e) Subtract 3 from both sides. (To undo 'add 3', apply the transformation 'subtract 3'.)

(f) Subtract 4 from both sides. (To undo 'add 4', apply the transformation 'subtract 4'.)

7. (a)

(1)

$$\begin{gathered} x - 1 = 4 \\ x - 1 + 1 = 4 + 1 \\ x = 5 \end{gathered}$$

(2)

$$\begin{array}{r} x - 1 = 4 \\ +1 \quad +1 \\ \hline x = 5 \end{array}$$

(3)

$$\begin{gathered} x - 1 = 4 \\ x = 5 \end{gathered} \quad \text{add 1}$$

(4)

$$\begin{gathered} x - 1 = 4 \\ x = 5 \end{gathered}$$

(5)

For all real numbers x,

$$x - 1 = 4 \iff x = 5\,.$$

(b)

(1)

$$\begin{gathered} 5 = 3 + x \\ 5 - 3 = 3 + x - 3 \\ 2 = x \\ x = 2 \end{gathered}$$

(2)

$$\begin{array}{r} 5 = 3 + x \\ -3 \quad -3 \\ \hline 2 = x \\ x = 2 \end{array}$$

(3)

$$\begin{gathered} 5 = 3 + x \\ 2 = x \\ x = 2 \end{gathered} \quad \text{subtract 3}$$

(4)

$$\begin{gathered} 5 = 3 + x \\ 2 = x \\ x = 2 \end{gathered}$$

(5)

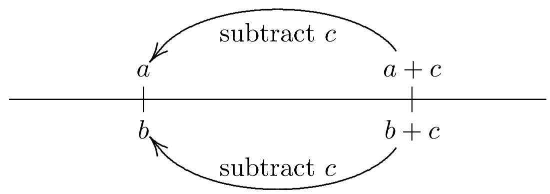
For all real numbers x,

$$\begin{aligned} 5 &= 3 + x \\ &\iff 2 = x \\ &\iff x = 2\,. \end{aligned}$$

8. (a)

- Suppose that '$a + c = b + c$' is true. Then, $a + c$ and $b + c$ live at the same place on a number line.
- Subtracting c from both numbers moves them both by the same amount.
- Since they started at the same place, they'll end up at the same place. Thus, '$a = b$' is also true.

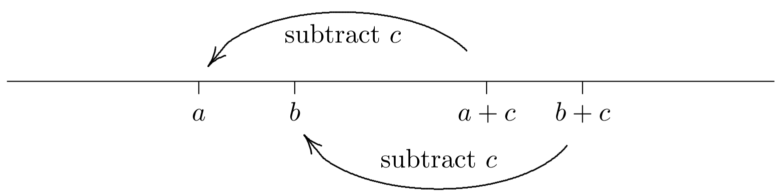
The sketch below assumes that c is a positive number:



(b)

- Suppose that '$a + c = b + c$' is false. Then, $a + c$ and $b + c$ live at different places on a number line.
- Subtracting c from both numbers moves them both by the same amount.
- Since they started at different places, they'll end up at different places. Thus, '$a = b$' is also false.

The sketch below assumes that c is a positive number:



END-OF-SECTION EXERCISES:

For problems 9—12, only one format is illustrated for each solution.

9.
$$\begin{aligned} 4x &= 5 + 3x \\ 4x - 3x &= 5 + 3x - 3x \\ x &= 5 \end{aligned}$$

CHECK:
$$\begin{aligned} 4(5) &\stackrel{?}{=} 5 + 3(5) \\ 20 &\stackrel{?}{=} 5 + 15 \\ 20 &= 20 \end{aligned}$$

10.
$$\begin{aligned} x + 4 &= 2x - 1 && \text{subtract } x \\ 4 &= x - 1 && \text{add } 1 \\ 5 &= x \\ x &= 5 \end{aligned}$$

CHECK:
$$\begin{aligned} 5 + 4 &\stackrel{?}{=} 2(5) - 1 \\ 9 &\stackrel{?}{=} 10 - 1 \\ 9 &= 9 \end{aligned}$$

11.
$$\begin{aligned} 3t - 7 + 2t &= 4t + 5 \\ 5t - 7 &= 4t + 5 \\ t - 7 &= 5 \\ t &= 12 \end{aligned}$$

CHECK:
$$\begin{aligned} 3(12) - 7 + 2(12) &\stackrel{?}{=} 4(12) + 5 \\ 36 - 7 + 24 &\stackrel{?}{=} 48 + 5 \\ 29 + 24 &\stackrel{?}{=} 53 \\ 53 &= 53 \end{aligned}$$

12.
$$\begin{aligned} 1 &= 8 - t && \text{add } t \\ 1 + t &= 8 && \text{subtract } 1 \\ t &= 7 \end{aligned}$$

CHECK:
$$\begin{aligned} 1 &\stackrel{?}{=} 8 - 7 \\ 1 &= 1 \end{aligned}$$