28. INTRODUCTION TO FACTORING

a mental game

Here's a mental game. I'm thinking of two numbers. When I multiply these numbers together, I get zero. Can you tell me anything about the numbers that I'm thinking of?

Answer: Yes! At least one of the numbers must be zero. The only way that two numbers can multiply to give 0 is for at least one of the factors to be 0. This is an incredibly useful mathematical fact. Here's the precise statement:

Zero Factor Law

Let a and b be real numbers. Then,

$$ab = 0 \quad \text{if and only if} \quad (a = 0 \text{ or } b = 0)\,.$$

Notice the appearance of the mathematical words 'if and only if' and 'or' in the statement of the Zero Factor Law. The exercises below review your understanding of these words.

EXERCISES

1. a. Suppose that the sentence '$S1$ if and only if $S2$' is true. What can be said about the truth values of the sentences $S1$ and $S2$?
   b. Suppose that the sentence 'A or B' is true. What can be said about the truth values of the sentences A and B?
   c. Suppose that $a = 0$ and $b = 3$. Is the sentence '$a = 0$ or $b = 0$' true or false?
   d. Suppose that $a = 3$ and $b = 0$. Is the sentence '$a = 0$ or $b = 0$' true or false?
   e. Suppose that $a = 0$ and $b = 0$. Is the sentence '$a = 0$ or $b = 0$' true or false?
   f. Suppose that $a = 3$ and $b = 4$. Is the sentence '$a = 0$ or $b = 0$' true or false?
   g. What are the best words to use to describe the situation where the sentence '$a = 0$ or $b = 0$' is true?

translation of the Zero Factor Law

Here's the translation of the Zero Factor Law, which reviews and reinforces language concepts studied in the previous section.

The Zero Factor Law is a sentence of the form:

$S1$	if and only if	$S2$
$ab = 0$	if and only if	$(a = 0 \text{ or } b = 0)$

Notice that:

$S1$ is the sentence '$ab = 0$'

$S2$ is the sentence '$(a = 0 \text{ or } b = 0)$'

forward and reverse directions of '$S1$ iff $S2$'

When a sentence of the form '$S1$ iff $S2$' is true, then the two subsentences ($S1$ and $S2$) must have the same truth values: they are both true, or both false. Focusing for the moment on the subsentences being *true*, we can in particular say the following:

- If $S1$ is true, then $S2$ is also true. This is often called the *forward direction*.
- If $S2$ is true, then $S1$ is also true. This is often called the *reverse direction*.

These two 'directions' are discussed next.

forward direction:

if $S1$ is true, then so is $S2$

forward direction:

if $S1$ is true, then so is $S2$;

if '$ab = 0$' is true, then so is '$(a = 0$ or $b = 0)$'

If the sentence '$ab = 0$' is true, then so is the sentence '$a = 0$ or $b = 0$'. Thus, if you have any two numbers that multiply to zero, then at least one of these two numbers must equal zero. This is the most useful direction of the 'if and only if' statement.

reverse direction:

If $S2$ is true, then so is $S1$

reverse direction:

if $S2$ is true, then so is $S1$;

if '$(a = 0$ or $b = 0)$' is true, then so is '$ab = 0$'

If the sentence '$a = 0$ or $b = 0$' is true, then so is the sentence '$ab = 0$'. If at least one of the factors in a multiplication problem is zero, then the product is zero. This direction of the 'if and only if' statement is certainly true, but is nowhere near as interesting or as useful as the forward direction.

the reason why having a product is so useful

The Zero Factor Law is the reason why having an expression written as a product (i.e., as a multiplication problem) is so useful. When a product is zero, we are able to say something about the factors.

EXERCISES

2. a. Suppose that the sentence $ab = 0$ is true. What (if anything) can be said about a and b?
   b. Suppose that the sentence $abcd = 0$ is true. What (if anything) can be said about a, b, c, and d?
   c. Suppose that the sentence $a + b = 0$ is true. What (if anything) can be said about a and b? In particular, must a or b equal zero?
   d. Suppose that the sentence $a+b+c+d = 0$ is true. What (if anything) can be said about a, b, c, and d? In particular, must one of the variables equal zero?
   e. If a product equals zero, then can anything be said about the factors?
   f. If a sum equals zero, then can anything be said about the numbers being added?

recognizing products:

the LAST operation is multiplication

A *product* is an expression where the *last* operation done is multiplication. This idea is illustrated with a few examples.

EXAMPLE

$a(b + c)$ is a product:

the last operation is multiplication

Consider the expression $a(b + c)$. If numbers are chosen for a, b, and c, then here is the order that computations would be done:

- Add b and c.
- **Multiply** this sum by a.

Notice that the *last* operation done is multiplication. Thus, the expression $a(b + c)$ is a product.

EXAMPLE

$(x - 3)(x + 2)$ is a product:

the last operation is multiplication

Here's another example. Consider the expression $(x-3)(x+2)$. Given a number x, here is the order that computations would be done:

- Subtract 3 from x. Set this aside.
- Add 2 to x. Set this aside.
- **Multiply** together the results from the previous two steps.

Notice again that the *last* operation done is multiplication. Thus, the expression $(x - 3)(x + 2)$ is a product.

EXAMPLE

$ab+c$ is NOT a product:

the last operation is NOT multiplication

As a last example, consider the expression $ab+c$. Given numbers a, b, and c, here is the order that computations would be done:

- Multiply a and b.
- **Add** this result to c.

Notice that the *last* operation done is addition, *not* multiplication. Thus, $ab+c$ is *not* a product. An expression where the last operation is addition is called a *sum*.

EXERCISES

3. Indicate the *last* operation that would be done in computing each of the following expressions. Then, identify each expression as either a *product* or a *sum*.
   a. xy
   b. $xy(z-1)$
   c. x^2+y^2
   d. $3x(x+2)$
   e. $x+3y$
   f. $2x(x-3)(x+1)$
   g. $x-2xy$
4. Consider the distributive law: $a(b+c)=ab+ac$. On one side of this equation there is a sum, and on one side there is a product. Which is which?

Here are two important definitions:

DEFINITIONS:

product; factors

sum; terms

A *product* is an expression where the last operation is multiplication. In a product, the things being multiplied are called the *factors*.

A *sum* is an expression where the last operation is addition. In a sum, the things being added are called the *terms*.

EXAMPLES

identifying factors and terms

Example: In the product $2xy$, the factors are 2, x, and y.

Example: In the product $(x+1)(x-2)$, the factors are $x+1$ and $x-2$.

Example: In the product $3x^2(2x+1)$, the factors are 3, x^2, and $2x+1$.

Example: In the sum $xy+3$, the terms are xy and 3. There are two terms.

Example: In the sum $5x^2+2x+4$, the terms are $5x^2$, $2x$, and 4. There are three terms.

EXAMPLE

a term includes its sign

Example: Remember that every subtraction problem is an addition problem in disguise. Suppose you're given the sum $x - y + 2xy - 1$, and asked to identify the terms. First, you must think of each subtraction as a special kind of addition:

$$x - y + 2xy - 1 = x + (-y) + 2xy + (-1)\,.$$

Then, you can identify the terms: they are x, $-y$, $2xy$, and -1. There are four terms.

Notice that each term 'includes' the plus or minus sign that precedes it (although a plus sign doesn't need to be written down). The phrase that people use to describe this fact is: *a term includes its sign.*

You should be able to identify the terms in a sum without actually having to rewrite it:

$$\overbrace{x}^{\text{term}}\ \overbrace{-y}^{\text{term}}\ \overbrace{+\ 2xy}^{\text{term}}\ \overbrace{-1}^{\text{term}}$$

EXAMPLE

Example: In the sum $2x^2y - 4xy + y^3 - 3x + 1$, the terms are $2x^2y$, $-4xy$, y^3, $-3x$, and 1. There are five terms.

EXERCISES

5. Identify each expression as either a product or a sum.
   In each product, identify the factors.
   In each sum, identify the terms.
   a. $3tx$
   b. $5x(x + 1)$
   c. $2x + 5x^2$
   d. $4(x + 1)(2x - 3)$
   e. $2 - 3xy + y^2 - x^2$
   f. $5xy - 2$

using the distributive law backwards

Consider the distributive law, $a(b+c) = ab+ac$. Rewrite this law 'backwards' (that is, from right to left) as

$$ab + ac = a(b + c)\,.$$

In this form, the distributive law provides a useful tool for taking a sum and writing it as a product, as discussed next. We used this direction of the distributive law in an earlier section, when talking about combining like terms. We'll use this direction again to talk about *factoring an expression*.

factoring an expression

To *factor an expression* means to take the expression and rename it as a product. That is, to *factor an expression* means to write the expression as a product.

EXAMPLE

factoring an expression; factoring $ab + ac$

For example, taking $ab+ac$ (a sum) and writing it as $a(b+c)$ (a product) is called *factoring*. We took the expression $ab+ac$ and renamed it as the product $a(b+c)$. Techniques for factoring expressions like $ab+ac$ are studied in this section and the next.

EXAMPLE

factoring an expression; factoring $x^2 - x - 2$

Here's another example. In an earlier section, you used FOIL to show that $(x+1)(x-2) = x^2 - x - 2$. This equation is true for all real numbers x. Writing the equation 'backwards' gives:

$$x^2 - x - 2 = (x+1)(x-2)$$

The process of going from the sum $x^2 - x - 2$ to the product $(x+1)(x-2)$ is called *factoring the expression* $x^2 - x - 2$. Techniques for factoring expressions like $x^2 - x - 2$ will be studied in a future section.

So, here's another important definition:

DEFINITION:

to factor an expression

To *factor an expression* means to rename the expression as a product.

using the distributive law to factor expressions of the form $ab + ac$

The distributive law is used to factor expressions of the form $ab + ac$. The key ideas are outlined below.

The expression $\mathbf{a}b$ has factors $\mathbf{a}$ and b.

The expression $\mathbf{a}c$ has factors $\mathbf{a}$ and c.

The factor '$\mathbf{a}$' is common to both terms; it is called a *common factor*. Notice how this common factor is underlined below:

$$\underline{a}b + \underline{a}c = \underline{a}(b+c)\,.$$

When the factor $\mathbf{a}$ is 'removed' from $\mathbf{a}b$, you are left with b.

When the factor $\mathbf{a}$ is 'removed' from $\mathbf{a}c$, you are left with c.

In going from the name $\mathbf{a}b + \mathbf{a}c$ to the name $\mathbf{a}(b+c)$, the common factor is first identified, and written down. Next, an opening parenthesis '(' is inserted. Then, the remaining parts of each term are written down. Finally, the closing parenthesis ')' is inserted.

two skills essential to the technique of factoring

Similarly, the expression $ab - ac$ is factored as $a(b-c)$:

$$\underline{a}b - \underline{a}c = \underline{a}(b-c)\,.$$

There are two skills essential to the technique of factoring:

- You must be able to recognize a product and identify the factors in the product.
- When you have two or more products, you must be able to recognize the common factor(s).

The concepts of *common factor* and *greatest common factor* are studied in the next section. This section is concluded with some simple examples of using the distributive law 'backwards' to factor.

EXAMPLES

using the distributive law 'backwards' to factor simple expressions

In each example below, notice how the common factor(s) are underlined. Always write the common factor(s) down first. Then, open the parentheses and write down what's left.

Example: $\underline{c}d - \underline{c}e = \underline{c}(d-e)$

Example: $x\underline{y} - \underline{y}z = \underline{y}(x-z)$

Example: $\underline{2x}t - \underline{2x}w = \underline{2x}(t-w)$

Example: $3\underline{x^2}t - y\underline{x^2} = \underline{x^2}(3t-y)$

Example: $2w\underline{(x+y)} - t\underline{(x+y)} = \underline{(x+y)}(2w-t)$

EXERCISES

6. Underline the common factor(s) in each expression below. Then, rename each expression as a product.
   a. $3x + 3t$
   b. $xy - zx$
   c. $5ab - 3bw$
   d. $2x^2y + 2x^2z$
   e. $3(x + 2) - 2t(x + 2)$
   f. $xyz - 5zy$

EXERCISES
web practice

Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTION TO EXERCISES: FACTORING

1. a. $S1$ and $S2$ have the same truth values; they're either both true, or both false

b. Either A is true, or B is true, or both A and B are true. That is, at least one of A or B is true.

c. true

d. true

e. true

f. false

g. You can say either of these:

a is 0, or b is zero, or both a and b are zero; OR

at least one of a or b is 0.

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2. a. either $a = 0$, or $b = 0$, or both a and b are 0; that is, at least one of a or b is 0

b. At least one of a or b or c or d is zero

c. If $a + b = 0$, then nothing can be said about a or b. Notice that $-3 + 3 = 0$; in particular, it might be that both a and b are not zero.

d. Notice that $3 + 7 + (-4) + (-6) = 0$. It is possible for $a + b + c + d$ to equal 0, but for all of the variables to be nonzero.

e. If a product is zero, then at least one of the factors must equal zero.

f. If a sum is zero, then nothing can be said about the numbers being added. In particular, it might be that all of the numbers being added are nonzero.

3. a. xy : last operation is multiplication; a product

b. $xy(z-1)$: last operation is multiplication; a product

c. $x^2 + y^2$: last operation is addition; a sum

d. $3x(x+2)$: last operation is multiplication; a product

e. $x + 3y$: last operation is addition; a sum

f. $2x(x-3)(x+1)$: last operation is multiplication; a product

g. $x - 2xy$: last operation is addition (subtraction is a special kind of addition); a sum

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4. $a(b+c)$ is a product; $ab + ac$ is a sum

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5. a. $3tx$ is a product; the factors are 3, t, and x

b. $5x(x+1)$ is a product; the factors are 5, x, and $x+1$

c. $2x + 5x^2$ is a sum; the terms are $2x$ and $5x^2$

d. $4(x+1)(2x-3)$ is a product; the factors are 4, $x+1$, and $2x-3$

e. $2 - 3xy + y^2 - x^2$ is a sum; the terms are 2, $-3xy$, y^2, and $-x^2$

f. $5xy - 2$ is a sum; the terms are $5xy$ and -2

6.

a. $\underline{3}x + \underline{3}t = \underline{3}(x+t)$

b. $\underline{x}y - z\underline{x} = \underline{x}(y-z)$

c. $5a\underline{b} - 3\underline{b}w = \underline{b}(5a-3w)$

d. $\underline{2x^2}y + \underline{2x^2}z = \underline{2x^2}(y+z)$

e. $3\underline{(x+2)} - 2t\underline{(x+2)} = \underline{(x+2)}(3-2t)$

f. $x\underline{yz} - 5\underline{zy} = \underline{yz}(x-5)$