

## 23. RATIONAL EXPONENTS

*renaming radicals*

When serious work needs to be done with radicals, they are usually changed to a name that uses exponents, so that the exponent laws can be used. Also, this new name for radicals will allow them to be approximated on any calculator that has a power key.

*rational numbers*

Recall that the *rational numbers* are ratios of integers. Any number that can be written in the form  $\frac{p}{q}$ , where  $p$  and  $q$  are integers and  $q \neq 0$ , is a rational number. Thus,  $\frac{1}{2}$ ,  $-\frac{3}{2} = \frac{-3}{2}$ , and  $5 = \frac{5}{1}$  are all rational numbers.

*writing radicals  
with rational exponents*

Here are the rational exponent names for radicals:

$$\begin{aligned}\sqrt{x} &= x^{\frac{1}{2}} \\ \sqrt[3]{x} &= x^{\frac{1}{3}} \\ \sqrt[4]{x} &= x^{\frac{1}{4}} \\ \sqrt[5]{x} &= x^{\frac{1}{5}} \\ \sqrt[6]{x} &= x^{\frac{1}{6}} \\ &\vdots \\ \sqrt[n]{x} &= x^{\frac{1}{n}}\end{aligned}$$

Regardless of the name used, the normal restrictions apply. For example,  $x^{\frac{1}{2}}$  is only defined for  $x \geq 0$ . Indeed, for any even number  $n$ ,  $x^{\frac{1}{n}}$  is only defined for nonnegative values of  $x$ .

*order of operation  
considerations:  
keying in  $8^{\frac{1}{3}}$   
correctly*

It's important to keep order of operation considerations in mind when you're keying rational exponent expressions into a calculator. In what follows, let  $\wedge$  denote 'to the power of'. Many calculators have a key like this, that is used for arbitrary exponents.

Suppose you want to compute  $8^{\frac{1}{3}}$  on a calculator. (Since  $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$ , you should of course get 2 as the result.) If you just key in

$$8 \wedge 1 \div 3$$

(try it!) you'll get approximately 2.667. What's happening here? Because exponentiation ( $\wedge$ ) is 'stronger' than division, here's what's happening:

$$8 \wedge 1 \div 3 = (8 \wedge 1) \div 3 = \frac{8^1}{3} \approx 2.667,$$

which is not what you want at all. Thus, you must use parentheses to force the division to be done first, and key in:

$$8 \wedge (1 \div 3)$$

which yields the correct result.

*'opposite'  
versus  
subtraction*

Some calculators (like the TI-83) have different keys for finding the opposite of a number and for subtraction.

On the TI-83, the 'find the opposite' key looks like a subtraction sign enclosed in parentheses: (-)

The subtraction key looks like a normal subtraction sign: -

To key in, say,  $-3 - 4$ , you'd use these keystrokes:

$$-3 - 4 \stackrel{\text{key in}}{=} (-) 3 - 4$$

In particular, something like ' $-2^3$ ' is the opposite of  $2^3$ . Keep this in mind as you read the next paragraph.

*keying in  
 $(-4)^{\frac{1}{2}}$  and  $-4^{\frac{1}{2}}$   
correctly*

Recall that  $(-4)^{\frac{1}{2}} = \sqrt{-4}$  (which is not defined) and  $-4^{\frac{1}{2}} = -\sqrt{4} = -2$  represent different orders of operations. They must be keyed in differently.

To key in  $(-4)^{\frac{1}{2}}$ , you could do this:

$$(-4)^{\frac{1}{2}} \stackrel{\text{key in}}{=} ( \overbrace{(-)}^{\text{opposite}} 4 ) \wedge ( 1 \div 2 )$$

(You should get some kind of an error, because this is not a real number.)

To key in  $-4^{\frac{1}{2}}$ , you could do this:

$$-4^{\frac{1}{2}} \stackrel{\text{key in}}{=} \overbrace{(-)}^{\text{opposite}} 4 \wedge ( 1 \div 2 )$$

(You should get  $-2$  as the result.)

Of course, different calculators have different keys and different options for computing exponents, so things may be done differently for you. But always be sure to keep order of operation considerations in mind when you're working with a calculator.

### EXERCISES

1. Write each radical in rational exponent form.
  - a.  $\sqrt[9]{x}$
  - b.  $\sqrt{5}$
  - c.  $\sqrt[127]{x+1}$
  - d.  $\sqrt[n]{x}$
2. Use the power key on your calculator to evaluate each of the following. Rewrite as a radical, and verify that you get the expected result.
  - a.  $16^{\frac{1}{2}}$
  - b.  $(-8)^{\frac{1}{3}}$
  - c.  $(-16)^{\frac{1}{4}}$
  - d.  $-16^{\frac{1}{4}}$

**EXERCISES**

3. Write each rational exponent expression as a radical. Simplify, if possible.
- $27^{\frac{1}{3}}$
  - $(-27)^{\frac{1}{3}}$
  - $-1^{\frac{1}{2048}}$
  - $(-1)^{\frac{1}{2048}}$
  - $8^{-\frac{1}{3}}$
  - $16^{(-1/4)}$

*making sense of  $8^{\frac{2}{3}}$*

Now, it is possible to define expressions with arbitrary rational number exponents. The idea is a simple application of an exponent law.

Suppose you need to evaluate  $8^{\frac{2}{3}}$ . You could write:

$$8^{\frac{2}{3}} = 8^{2 \cdot \frac{1}{3}} = (8^2)^{\frac{1}{3}} = 64^{\frac{1}{3}} = \sqrt[3]{64} = 4$$

Or, you could write:

$$8^{\frac{2}{3}} = 8^{\frac{1}{3} \cdot 2} = (8^{\frac{1}{3}})^2 = (\sqrt[3]{8})^2 = 2^2 = 4$$

Both ways are correct. Either route gets you to the same answer. The second way is certainly easier, because it leads to the evaluation of  $\sqrt[3]{8}$ , which is much easier than the evaluation of  $\sqrt[3]{64}$ .

*evaluating  $x^{\frac{p}{q}}$*

As long as everything is defined,

$$x^{\frac{p}{q}} = (x^p)^{\frac{1}{q}} = \sqrt[q]{x^p}$$

or

$$x^{\frac{p}{q}} = (x^{\frac{1}{q}})^p = (\sqrt[q]{x})^p.$$

In both cases, the denominator in the exponent indicates the type of root.

**EXAMPLES**

Write each rational exponent expression as a radical in two different ways. Evaluate each resulting radical, and decide which is easier.

$16^{\frac{3}{4}}$

Example:  $16^{\frac{3}{4}}$

$$16^{\frac{3}{4}} = (16^3)^{\frac{1}{4}} = \sqrt[4]{16^3} = \sqrt[4]{4096} = 8$$

$$16^{\frac{3}{4}} = (16^{\frac{1}{4}})^3 = (\sqrt[4]{16})^3 = 2^3 = 8$$

The second way is much easier.

$32^{\frac{6}{5}}$

Example:  $32^{\frac{6}{5}}$

$$32^{\frac{6}{5}} = (32^6)^{\frac{1}{5}} = \sqrt[5]{32^6} = \sqrt[5]{1,073,741,824} = 64$$

$$32^{\frac{6}{5}} = (32^{\frac{1}{5}})^6 = (\sqrt[5]{32})^6 = 2^6 = 64$$

The second way is *much* easier.

**EXERCISES**

4. Write each rational exponent expression as a radical in two different ways. Evaluate each resulting radical, and decide which is easier.
- $8^{\frac{7}{3}}$
  - $64^{\frac{5}{6}}$

*three different approaches;  
three different answers*

Here is a subtle and somewhat bothersome situation that can arise with rational exponents.

Let's evaluate  $(-8)^{\frac{2}{6}}$  in several ways. First:

$$(-8)^{\frac{2}{6}} = ((-8)^{\frac{1}{6}})^2 = (\sqrt[6]{-8})^2;$$

We can't continue, because  $\sqrt[6]{-8}$  is an even root of a negative number, which is not defined.

Here's a second way:

$$(-8)^{\frac{2}{6}} = (-8)^{\frac{1}{3}} = \sqrt[3]{-8} = -2$$

Here's a third way:

$$(-8)^{\frac{2}{6}} = ((-8)^2)^{\frac{1}{6}} = (64)^{\frac{1}{6}} = \sqrt[6]{64} = 2$$

Whoa. Three different (all seemingly correct) approaches. Three different answers. What's going on here?

*What does the calculator say?*

Let's see what the calculator says. The author has two calculators on her desktop. She evaluated  $(-8)^{\frac{2}{6}}$  on both, and got two different answers.

*here's how we're going to deal with this situation*

The way we're going to deal with this situation is to avoid it.

If  $x > 0$ , then  $x^{\frac{p}{q}}$  will never cause any problems. The fraction can be positive or negative; in simplest form or not;  $q$  can be even or odd. With  $x$  positive, all is right in the mathematical world.

Otherwise, we'll impose two requirements:

- We will only evaluate expressions of the form  $x^{\frac{p}{q}}$  when the fraction is in simplest form. (Note that  $\frac{2}{6}$  is *not* in simplest form.)
- Also, if we're considering  $x^{\frac{p}{q}}$  where  $q$  is an even number, then  $x$  must be nonnegative.

And, as usual, any expression that results in division by zero is undefined.

With these restrictions, you can rest assured that no matter how you use the exponent laws (correctly), you'll always get to the same place. Therefore, you can focus your attention on doing things in the simplest way.

*if you're clever . . .*

If you're clever, then you can take a problem that looks like it will require a calculator, rename it, and solve it without a calculator. Here's an example:

$$\sqrt[6]{64^5} = (\sqrt[6]{64})^5 = 2^5 = 32$$

Keep this in mind as you study the following examples.

## EXAMPLES

Write each radical as a rational exponent expression, and then evaluate it. Use your calculator as needed. If no exact answer is possible, then round your answer to six decimal places.

Example:  $\sqrt[6]{15625} = (15625)^{\frac{1}{6}} = 5$

Example:  $\sqrt[3]{7^5} = (7^5)^{\frac{1}{3}} = 7^{\frac{5}{3}} \approx 25.615140$

Example:  $(\sqrt[5]{16807})^3 = ((16807)^{\frac{1}{5}})^3 = (16807)^{\frac{3}{5}} = 343$

Example:  $\sqrt[5]{32^9} = (\sqrt[5]{32})^9 = 2^9 = 512$

**EXERCISES**

5. Write each radical as a rational exponent expression, and then evaluate it. Use your calculator as needed. If no exact answer is possible, then round your answer to six decimal places.

a.  $\sqrt[9]{512}$

b.  $\sqrt[9]{-512}$

c.  $\sqrt[6]{3^5}$

d.  $\sqrt[4]{16^7}$

**EXERCISES**

*web practice*

Go to my homepage <http://onemathematicalcat.org> and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

## SOLUTION TO EXERCISES: RATIONAL EXPONENTS

1. a.  $\sqrt[9]{x} = x^{\frac{1}{9}}$

b.  $\sqrt{5} = 5^{\frac{1}{2}}$

c.  $\sqrt[127]{x+1} = (x+1)^{\frac{1}{127}}$

d.  $\sqrt[m]{x} = x^{\frac{1}{m}}$

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2. a.  $16^{\frac{1}{2}} \stackrel{\text{key in}}{=} 16 \wedge (1 \div 2) = 4$ ;  $16^{\frac{1}{2}} = \sqrt{16} = 4$

b.  $(-8)^{\frac{1}{3}} \stackrel{\text{key in}}{=} ((-8) \wedge (1 \div 3)) = -2$ ;  $(-8)^{\frac{1}{3}} = \sqrt[3]{-8} = -2$

c.  $(-16)^{\frac{1}{4}} \stackrel{\text{key in}}{=} ((-16) \wedge (1 \div 4))$  which should produce an error;  $(-16)^{\frac{1}{4}} = \sqrt[4]{-16}$  is not defined

d.  $-16^{\frac{1}{4}} \stackrel{\text{key in}}{=} (-16 \wedge (1 \div 4)) = -2$ ;  $-16^{\frac{1}{4}} = -\sqrt[4]{16} = -2$

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3. a.  $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

b.  $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3$

c.  $-1^{\frac{1}{2048}} = -\sqrt[2048]{1} = -1$

d.  $(-1)^{\frac{1}{2048}} = \sqrt[2048]{-1}$  is not defined

e.  $8^{-\frac{1}{3}} = \frac{1}{8^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{8}} = \frac{1}{2}$

f.  $16^{(-1/4)} = \frac{1}{16^{\frac{1}{4}}} = \frac{1}{\sqrt[4]{16}} = \frac{1}{2}$

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4. a.  $8^{\frac{7}{3}} = (8^7)^{\frac{1}{3}} = \sqrt[3]{8^7} = \sqrt[3]{2097152} = 128$

$8^{\frac{7}{3}} = (8^{\frac{1}{3}})^7 = (\sqrt[3]{8})^7 = 2^7 = 128$

The second way is definitely easier.

b.  $64^{\frac{5}{6}} = (64^5)^{\frac{1}{6}} = \sqrt[6]{64^5} = \sqrt[6]{1,073,741,824} = 32$

$64^{\frac{5}{6}} = (64^{\frac{1}{6}})^5 = (\sqrt[6]{64})^5 = 2^5 = 32$

The second way is *definitely* easier.

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5. a.  $\sqrt[9]{512} = 512^{\frac{1}{9}} = 2$

b.  $\sqrt[9]{-512} = (-512)^{\frac{1}{9}} = -2$

c.  $\sqrt[6]{3^5} = (3^5)^{\frac{1}{6}} = 3^{\frac{5}{6}} \approx 2.49805$

d.  $\sqrt[4]{16^7} = (\sqrt[4]{16})^7 = 2^7 = 128$