22. RADICALS

doing something, then undoing it

The concept of doing something and then 'undoing' it is very important in mathematics. Here are some examples:

Take a number. Add 5 to it. How can you get back to the original number? Answer: subtract 5. That is, addition is 'undone' with subtraction.

$$x \overset{\text{add } 5}{\longrightarrow} x+5 \overset{\text{subtract } 5}{\longrightarrow} x$$

Take a number. Multiply it by 7. How can you get back to the original number? Answer: divide by 7. Multiplication is 'undone' by division.

$$x \overset{\text{multiply by } 7}{\longrightarrow} 7x \overset{\text{divide by } 7}{\longrightarrow} x$$

undoing powers

In this section, we address the issue of 'undoing' powers, like this:

Take a number. Cube it—that is, raise it to the third power. How can you get back to the original number?

$$x \overset{\text{cube it}}{\longrightarrow} x^3 \overset{\text{How can we get back?}}{\longrightarrow} x$$

Questions like this will lead us to an understanding of a mathematical expression called a *radical*.

undoing a cube

Let's revisit the scenario in the previous paragraph.

Take the number 2. Cube it, to get 8. Now, think about the thought process needed to get back to the original number. You must think: What number, when cubed, gives 8? The answer is of course 2.

$$2 \overset{\text{cube it}}{\longrightarrow} 8 \overset{\text{What number, when cubed, gives 8?}}{\longrightarrow} 2$$

One more time. Take the number -2. Cube it, to get -8. Ask the question: What number, when cubed, gives -8? The answer is -2.

$$-2 \overset{\text{cube it}}{\longrightarrow} -8 \overset{\text{What number, when cubed, gives } -8\text{?}}{\longrightarrow} -2$$

the cube root of 8

Notice that there is *only one* number which, when cubed, gives 8. This unique number is denoted by the symbol $\sqrt[3]{8}$ and is called the *cube root of* 8.

Also, there is *only one* number which, when cubed gives -8. This unique number is denoted by $\sqrt[3]{-8}$ and is called the *cube root of* -8.

The "cube root" process undoes the "cube" process. Roots undo powers. The idea is stated more formally below.

DEFINITION
the cube root of x

Let x be any real number. The number $\sqrt[3]{x}$, read as 'the cube root of x,' is defined as follows:

$$\sqrt[3]{x} = \text{the unique number which, when cubed, equals } x\,.$$

numbers have lots of different names!

$\sqrt[3]{8} = 2$

Don't ever lose sight of the fact that numbers have lots of different names. Both '2' and '$\sqrt[3]{8}$' are names for the same number. Thus, the sentence '$\sqrt[3]{8} = 2$' is true.

a good visual check

There's a good visual check for problems like this. As shown below, make a circle, checking that your answer (2), raised to the 3rd power, does indeed equal 8.

$$2^3$$

$$\sqrt[3]{8} = 2$$

$$= 8$$

EXAMPLES

Read each of the following aloud. State the thought process needed to get a simpler name for the number. Write a complete sentence that summarizes the result.

$\sqrt[3]{27}$

Example: $\sqrt[3]{27}$

read aloud: the cube root of 27

thought process: What number, when cubed, gives 27? Answer: 3

summarize result: $\sqrt[3]{27} = 3$

$\sqrt[3]{0}$

Example: $\sqrt[3]{0}$

read aloud: the cube root of 0

thought process: What number, when cubed, gives 0? Answer: 0

summarize result: $\sqrt[3]{0} = 0$

$\sqrt[3]{-\frac{1}{8}}$

Example: $\sqrt[3]{-\frac{1}{8}}$

read aloud: the cube root of $-\frac{1}{8}$

thought process: What number, when cubed, gives $-\frac{1}{8}$? Answer: $-\frac{1}{2}$

summarize result: $\sqrt[3]{-\frac{1}{8}} = -\frac{1}{2}$

$\sqrt[3]{(1.7)^3}$

Example: $\sqrt[3]{(1.7)^3}$

read aloud: the cube root of $(1.7)^3$

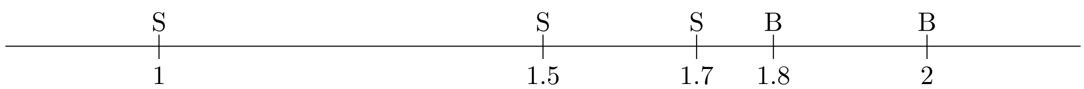
thought process: What number, when cubed, gives $(1.7)^3$? Answer: 1.7

summarize result: $\sqrt[3]{(1.7)^3} = 1.7$

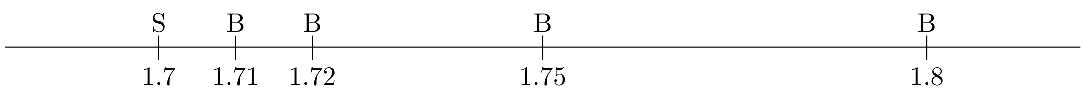
What number, when cubed, gives 5 ?

Suppose you're asked to find $\sqrt[3]{5}$. What number, when cubed, gives 5 ? Well, $1^3 = 1$, which isn't big enough. And, $2^3 = 8$, which is too big. Somewhere between 1 and 2 , there is a number which, when cubed, gives 5 . Let's try to find it. The results of the calculations below are summarized on number lines, with 'S' denoting too small, and 'B' denoting too big. Verify these figures yourself on your own calculator!

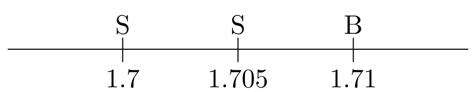
$(1.5)^3 = 3.375$	too small
$(1.8)^3 = 5.832$	too big; answer is between 1.5 and 1.8
$(1.7)^3 = 4.913$	too small; answer is between 1.7 and 1.8
$(1.75)^3 = 5.359375$	too big; answer is between 1.7 and 1.75
$(1.72)^3 = 5.088448$	too big; answer is between 1.7 and 1.72
$(1.71)^3 = 5.000211$	too big; answer is between 1.7 and 1.71
$(1.705)^3 = 4.956477625$	too small; answer is between 1.705 and 1.71



ZOOM IN:



ZOOM IN:



continuing the process . . .

This process could continue ad infinitum, with us getting closer and closer to $\sqrt[3]{5}$. There *is* a unique real number which, when cubed, gives 5 . Unfortunately, however, it doesn't have a very nice decimal name. (★ It is an infinite, non-repeating decimal.) So, how can we talk about this number? Using the name $\sqrt[3]{5}$!

using your calculator to approximate $\sqrt[3]{5}$

If you need a decimal approximation to $\sqrt[3]{5}$, you can use your calculator. Look for a $\sqrt[3]{\ }$ key, or ask your teacher how your calculator does cube roots. You should find that $\sqrt[3]{5} \approx 1.709975947$, which is indeed a number between 1.705 and 1.71 .

EXERCISES

1. Read each of the following aloud. State the thought process needed to get a simpler name for the number (if a simpler name exists). Write a complete sentence that summarizes the result. If no simpler name exists, first find two numbers that the cube root must lie between, and then use your calculator to find an approximation rounded to 5 decimal places.
   a. $\sqrt[3]{-27}$
   b. $\sqrt[3]{1}$
   c. $\sqrt[3]{-1}$
   d. $\sqrt[3]{11}$
   e. $\sqrt[3]{\frac{1}{27}}$
   f. $\sqrt[3]{-\frac{1}{1000}}$
   g. $\sqrt[3]{(2.735)^3}$
   h. $\sqrt[3]{-14}$

the cube root undoes the cubing operation

Note that the cube root operation undoes the cubing operation:

$$x \overset{\text{cube}}{\rightarrow} x^3 \overset{\text{take cube root}}{\rightarrow} x$$

In other words, for all real numbers x,

$$\sqrt[3]{x^3} = x\,.$$

$\sqrt[3]{\textit{positive}} = \textit{positive}$
$\sqrt[3]{\textit{negative}} = \textit{negative}$

Note also that the cube root of a positive number is positive, and the cube root of a negative number is negative.

The number 3 is an odd number, and $\sqrt[3]{\ }$ is an odd root. All odd roots are defined in the same way. (Even roots will be considered momentarily.) Here's the precise definition and some properties of odd roots:

DEFINITION
$\sqrt[n]{x}$, *for odd values of* n

Let x be any real number, and let $n \in \{3, 5, 7, 9, \dots\}$.
The number $\sqrt[n]{x}$ is defined as follows:

$$\sqrt[n]{x} = \text{the unique number which, when raised to the } n^{\text{th}} \text{ power, equals } x\,.$$

PROPERTIES OF ODD ROOTS

For all real numbers x and for $n \in \{3, 5, 7, 9, \dots\}$,

$$\sqrt[n]{x^n} = x$$
$$\text{if } x > 0\text{, then } \sqrt[n]{x} > 0$$
$$\text{if } x < 0\text{, then } \sqrt[n]{x} < 0$$

reading odd roots aloud

Here are examples of how to read odd roots. Note that $\sqrt[3]{x}$ has a special name, and that the endings 'st', 'rd' and 'th' correspond to the endings in the words fir**st**, thi**rd**, fif**th**, seven**th**, and nin**th**.

expression	how to read
$\sqrt[n]{x}$	the n^{th} root of x
$\sqrt[3]{x}$	the cube root of x
$\sqrt[5]{x}$	the 5^{th} root of x
$\sqrt[21]{x}$	the 21^{st} root of x
$\sqrt[23]{x}$	the 23^{rd} root of x
$\sqrt[25]{x}$	the 25^{th} root of x
$\sqrt[27]{x}$	the 27^{th} root of x
$\sqrt[29]{x}$	the 29^{th} root of x

for your convenience

It may help you with the problems in this section to have the following information at your fingertips:

$$\begin{array}{ll} 2^2 = 4 & 3^2 = 9 \\ 2^3 = 8 & 3^3 = 27 \\ 2^4 = 16 & 3^4 = 81 \\ 2^5 = 32 & 3^5 = 243 \\ 2^6 = 64 & \\ 2^7 = 128 & \end{array}$$

EXAMPLES

Read each of the following aloud. State the thought process needed to get a simpler name for the number. Write a complete sentence that summarizes the result.

$\sqrt[5]{32}$

Example: $\sqrt[5]{32}$

read aloud: the fifth root of 32

thought process: What number, when raised to the 5^{th} power, gives 32? Answer: 2

summarize result: $\sqrt[5]{32} = 2$

$\sqrt[5]{-32}$

Example: $\sqrt[5]{-32}$

read aloud: the fifth root of -32

thought process: What number, when raised to the 5^{th} power, gives -32? Answer: -2

summarize result: $\sqrt[5]{-32} = -2$

$\sqrt[219]{1}$

Example: $\sqrt[219]{1}$

read aloud: the 219^{th} root of 1

thought process: What number, when raised to the 219^{th} power, gives 1? Answer: 1

summarize result: $\sqrt[219]{1} = 1$

$\sqrt[219]{-1}$

Example: $\sqrt[219]{-1}$

read aloud: the 219^{th} root of -1

thought process: What number, when raised to the 219^{th} power, gives -1? Answer: -1

summarize result: $\sqrt[219]{-1} = -1$

$\sqrt[11]{(4.932)^{11}}$

Example: $\sqrt[11]{(4.932)^{11}}$

read aloud: the 11^{th} root of $(4.932)^{11}$

thought process: What number, when raised to the 11^{th} power, gives $(4.932)^{11}$? Answer: 4.932

summarize result: $\sqrt[11]{(4.932)^{11}} = 4.932$

EXERCISES

2. Read each of the following aloud. State the thought process needed to get a simpler name for the number. Write a complete sentence that summarizes the result.
   a. $\sqrt[7]{128}$
   b. $\sqrt[7]{-128}$
   c. $\sqrt[19]{1}$
   d. $\sqrt[19]{-1}$
   e. $\sqrt[571]{0}$
   f. $\sqrt[13]{(-3.9274)^{13}}$

even roots:

Next, we'll try to 'undo' even powers, and will see that two problems emerge.

the first problem

Before beginning, recall that $2^2 = 4$ and $(-2)^2 = 4$. That is, there are two numbers, both 2 and -2, which, when squared, give 4.

two numbers, when squared, give 4

So, suppose I tell you that I'm thinking of a number. When I square this number, I get 4. Can you tell me which number I'm thinking of? No—I could be thinking of the number 2, or the number -2. This is the first problem with even roots.

Ask: What nonnegative number, when squared, gives 4 ?

The question 'What number, when squared, gives 4 ? ' is flawed, because there is not a unique number with this property. We must instead ask a different question in order to get a unique answer: 'What nonnegative number, when squared, gives 4 ? ' Then, the answer is 2.

use the name $\sqrt{4}$, not $\sqrt[2]{4}$

The symbol '$\sqrt{4}$' is read as 'the square root of 4' and represents the *nonnegative* number which, when squared, gives 4. Recall that *nonnegative* means not negative; i.e., greater than or equal to zero. Notice that we don't use the symbol '$\sqrt[2]{4}$', which you may have expected. This root is given a special name (the square root) and a special symbol ($\sqrt{\ }$).

even roots:

the second problem

Next, suppose you are asked to find $\sqrt{-4}$. Is there *any* real number which, when squared, gives -4? A positive number, when squared, is positive. A negative number, when squared, is again positive. Thus, no real number exists which has the property that squaring it gives -4 as the result. This is the second problem. You can't take the square root of negative numbers.

★

Well, at least not when you're working in $\mathbb{R}$. In the complex numbers, if x is a negative real number, then $\sqrt{x} = i\sqrt{|x|}$.

Thus, we are led to the precise definition of the square root, which is characteristic of the behavior of all even roots:

DEFINITION
the square root of x

Let $x \geq 0$. The number $\sqrt{x}$, read as 'the square root of x,' is defined as follows:

$$\sqrt{x} = \text{the nonnegative number which, when squared, equals } x\,.$$

EXAMPLES

Read each of the following aloud. State the thought process needed to get a simpler name for the number. Write a complete sentence that summarizes the result.

$\sqrt{36}$

Example: $\sqrt{36}$

read aloud: the square root of 36

think: What nonnegative number, when squared, gives 36? Answer: 6

summarize result: $\sqrt{36} = 6$

$\sqrt{1}$

Example: $\sqrt{1}$

read aloud: the square root of 1

think: What nonnegative number, when squared, gives 1? Answer: 1

summarize result: $\sqrt{1} = 1$

$\sqrt{0}$

Example: $\sqrt{0}$

read aloud: the square root of 0

think: What nonnegative number, when squared, gives 0? Answer: 0

summarize result: $\sqrt{0} = 0$

$\sqrt{(7.92)^2}$

Example: $\sqrt{(7.92)^2}$

read aloud: the square root of $(7.92)^2$

think: What nonnegative number, when squared, gives $(7.92)^2$? Answer: 7.92

summarize result: $\sqrt{(7.92)^2} = 7.92$

$\sqrt{(-7)^2}$

Example: $\sqrt{(-7)^2}$

Be careful with this one!

read aloud: the square root of $(-7)^2$

think: What nonnegative number, when squared, gives $(-7)^2$?

Answer: The answer can't be -7, because it's negative. Is there any other number which, when squared, gives $(-7)^2$? Sure! $7^2 = (-7)^2$. So, the answer is 7.

summarize result: $\sqrt{(-7)^2} = 7$

Here's the generalization of the definition to cover all even roots:

DEFINITION
$\sqrt[n]{x}$,
for even values of n

Let $x \geq 0$, and let $n \in \{2, 4, 6, 8, \dots\}$.
The number $\sqrt[n]{x}$ is defined as follows:

$$\sqrt[n]{x} = \text{the nonnegative number which, when raised to the } n^{\text{th}} \text{ power, equals } x\,.$$

reading even roots aloud

Here are examples of how to read even roots. Note that the endings ' th ' and ' nd ' correspond to the endings in the words seco**nd**, four**th**, six**th**, eigh**th**, and ten**th**.

expression	how to read
$\sqrt[n]{x}$	the n^{th} root of x
$\sqrt{x}$	the square root of x
$\sqrt[4]{x}$	the 4^{th} root of x
$\sqrt[20]{x}$	the 20^{th} root of x
$\sqrt[22]{x}$	the 22^{nd} root of x
$\sqrt[24]{x}$	the 24^{th} root of x
$\sqrt[26]{x}$	the 26^{th} root of x
$\sqrt[28]{x}$	the 28^{th} root of x

EXAMPLES

Read each of the following aloud. State the thought process needed to get a simpler name for the number. Write a complete sentence that summarizes the result. If an expression is not defined, so state.

$\sqrt[4]{16}$

Example: $\sqrt[4]{16}$

read aloud: the fourth root of 16

thought process: What nonnegative number, when raised to the 4^{th} power, gives 16 ? Answer: 2

summarize result: $\sqrt[4]{16} = 2$

$\sqrt[368]{1}$

Example: $\sqrt[368]{1}$

read aloud: the 368^{th} root of 1

thought process: What nonnegative number, when raised to the 368^{th} power, gives 1 ? Answer: 1

summarize result: $\sqrt[368]{1} = 1$

$\sqrt[746]{(-9.467)^{746}}$

Example: $\sqrt[746]{(-9.467)^{746}}$

read aloud: the 746^{th} root of $(-9.467)^{746}$

thought process: What nonnegative number, when raised to the 746^{th} power, gives $(-9.467)^{746}$? Answer: 9.467 . Be careful!

summarize result: $\sqrt[746]{(-9.467)^{746}} = 9.467$

$\sqrt[4]{-16}$

Example: $\sqrt[4]{-16}$

read aloud: the fourth root of -16

thought process: What nonnegative number, when raised to the 4^{th} power, gives -16 ? Answer: no such number exists

summarize result: $\sqrt[4]{-16}$ is not defined *two different questions; two different answers:*

It's important to realize that there are two different questions you might be asked, which have two different answers.

Solve $x^2 = 4$.

Find $\sqrt{4}$.

FIRST QUESTION: Solve the equation $x^2 = 4$.

This means to find all possible numbers which, when substituted for x, make the equation true.

A number that makes the equation true is called a *solution* of the equation.

The number 2 is a solution, since $2^2 = 4$. The number -2 is also a solution, since $(-2)^2 = 4$.

Thus, the solutions to $x^2 = 4$ are $x = \pm 2$ (which is a shorthand for '$x = 2$ or $x = -2$').

SECOND QUESTION: Find $\sqrt{4}$.

The answer is $\sqrt{4} = 2$. The symbol $\sqrt{4}$ is asking for the *nonnegative* number which, when squared, gives 4.

EXERCISES

3. Solve the following equations. That is, find all possible numbers that make the equation true.
   a. $x^2 = 9$
   b. $x^2 = 1$
   c. $x^2 = 0$
   d. $x^2 = -9$
4. Simplify. If a number does not exist, so state.
   a. $\sqrt[6]{64}$
   b. $\sqrt[2468]{1}$
   c. $\sqrt[12]{(98)^{12}}$
   d. $\sqrt[12]{(-98)^{12}}$
   e. $\sqrt[n]{0}$, where n is a positive integer, $n \ge 2$
   f. $\sqrt[6]{-64}$

All even and odd roots go by a common name:

DEFINITION
radical

A *radical* is an expression of the form $\sqrt[n]{x}$, where $n \in \{2, 3, 4, \dots\}$.

The term *radical* refers to a particular name for a number. Both $\sqrt{4}$ and 2 are names for the same number: $\sqrt{4}$ is called a radical, but 2 isn't. You must see the root symbol $\sqrt{\ }$ for an expression to be called a radical.

simplifying a radical: even or odd root?

Whenever you're presented with a radical, like $\sqrt[5]{32}$ or $\sqrt[4]{16}$, you must first decide if you're dealing with an odd root or an even root. Odd roots are defined for all real numbers, but even roots are only defined for nonnegative numbers. Odd roots can give nonnegative or negative answers, but even roots only give nonnegative answers, when they exist.

using your calculator to approximate radicals

When a radical doesn't have a simple name, then your calculator can be used to get an approximation. You've probably located the square root ($\sqrt{\ }$) and cube root ($\sqrt[3]{\ }$) keys on your calculator. But, what about other roots? The answer comes in the next section, where we will study rational exponents, like $x^{\frac{1}{5}}$.

EXERCISES *web practice*	Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTION TO EXERCISES: RADICALS

1. a. read aloud: the cube root of -27

thought process: What number, when cubed, is -27 ? Answer: -3

summarize result: $\sqrt[3]{-27} = -3$

b. read aloud: the cube root of 1

thought process: What number, when cubed, is 1 ? Answer: 1

summarize result: $\sqrt[3]{1} = 1$

c. read aloud: the cube root of -1

thought process: What number, when cubed, is -1 ? Answer: -1

summarize result: $\sqrt[3]{-1} = -1$

d. read aloud: the cube root of 11

thought process: What number, when cubed, is 11 ? Answer: no nice number

$2^3 = 8$ (too small); $3^3 = 27$ (too big); so $\sqrt[3]{11}$ lies between 2 and 3 .

calculator approximation: $\sqrt[3]{11} \approx 2.22398$

Be sure to use the 'approximately equal to' verb.

e. read aloud: the cube root of $\frac{1}{27}$

thought process: What number, when cubed, is $\frac{1}{27}$? Answer: $\frac{1}{3}$

summarize result: $\sqrt[3]{\frac{1}{27}} = \frac{1}{3}$

f. read aloud: the cube root of $-\frac{1}{1000}$

thought process: What number, when cubed, is $-\frac{1}{1000}$? Answer: $-\frac{1}{10}$

summarize result: $\sqrt[3]{-\frac{1}{1000}} = -\frac{1}{10}$

g. read aloud: the cube root of -14

thought process: What number, when cubed, is -14 ? Answer: no nice number

$(-2)^3 = -8$; $(-3)^3 = -27$; so $\sqrt[3]{-14}$ lies between -2 and -3 .

calculator approximation: $\sqrt[3]{-14} \approx -2.41014$

2. a. read aloud: the 7^{th} root of 128

thought process: What number, when raised to the 7^{th} power, is 128 ? Answer: 2

summarize result: $\sqrt[7]{128} = 2$

b. read aloud: the 7^{th} root of -128

thought process: What number, when raised to the 7^{th} power, is -128 ? Answer: -2

summarize result: $\sqrt[7]{-128} = -2$

c. read aloud: the 19^{th} root of 1

thought process: What number, when raised to the 19^{th} power, is 1 ? Answer: 1

summarize result: $\sqrt[19]{1} = 1$

d. read aloud: the 19^{th} root of -1

thought process: What number, when raised to the 19^{th} power, is -1 ? Answer: -1

summarize result: $\sqrt[19]{-1} = -1$

e. read aloud: the 571^{st} root of 0

thought process: What number, when raised to the 571^{st} power, is 0 ? Answer: 0

summarize result: $\sqrt[571]{0} = 0$

f. read aloud: the 13^{th} root of $(-3.9274)^{13}$

thought process: What number, when raised to the 13^{th} power, is $(-3.9274)^{13}$? Answer: -3.9274

summarize result: $\sqrt[13]{(-3.9274)^{13}} = -3.9274$

3. a. $x^2 = 9$ has solutions $x = \pm 3$

b. $x^2 = 1$ has solutions $x = \pm 1$

c. $x^2 = 0$ has solution $x = 0$

d. $x^2 = -9$ has no real number solutions

4. a. $\sqrt[6]{64} = 2$

b. $\sqrt[2468]{1} = 1$

c. $\sqrt[12]{(98)^{12}} = 98$

d. $\sqrt[12]{(-98)^{12}} = 98$

e. If n is any positive integer greater than or equal to 2, then $\sqrt[n]{0} = 0$.

f. $\sqrt[6]{-64}$ does not exist