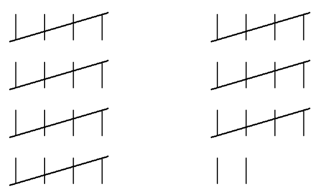
7. THE BASE TEN NUMBER SYSTEM

wonderfully efficient system for representing numbers

The system that we use to represent numbers is wonderfully efficient and simple. Large numbers can be represented with very few symbols in a neat and organized way. The method rests on two concepts, *base* and *place value*, which are briefly discussed in the next few paragraphs. The concepts are introduced using base three for simplicity. Then, the concepts are extended to our base ten number system.

tally mark representation | base ten representation



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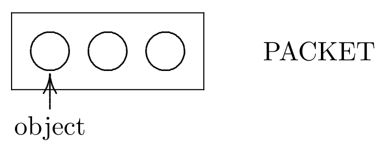
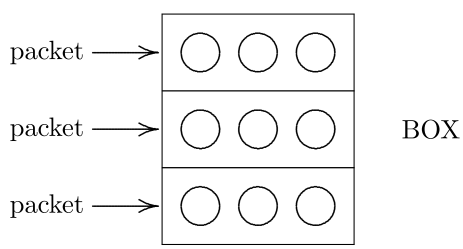
concept of 'base'

The concept of *base* is best represented by grouping. Imagine a factory where objects are being packaged for shipment around the world. Workers are instructed to 'bundle things up' every time they see a group of three. Here's what happens:

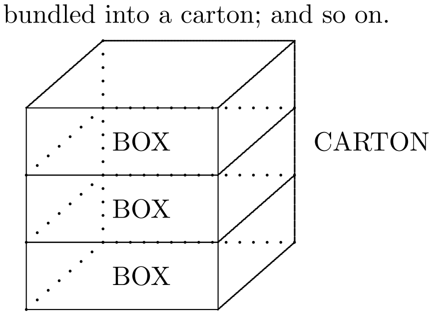
- three objects are bundled into a packet;



- three packets are bundled into a box;



- three boxes are bundled into a carton; and so on.



Notice that a packet holds 3 objects. A box holds three packets, hence $3 \cdot 3 = 9$ objects. A carton holds three boxes, hence $3 \cdot 9 = 27$ objects.

How would 46 *objects get bundled?*

Suppose a shipment of 46 objects is to go out to Pittsfield, Massachusetts. How would these 46 objects get bundled?

$$
\begin{array}{rl}
46 & \\
-\ 27 & \text{1 CARTON} \\
\hline
19 & \\
-\ 18 & \text{2 BOXES} \\
\hline
1 & \text{1 OBJECT}
\end{array}
$$

As the computation above shows, 46 objects would get bundled into 1 carton and 2 boxes, with 1 object left over.

EXERCISES

1. In the factory described above, how would 31 objects get bundled?
2. In the factory described above, how would 68 objects get bundled?

keeping track of the bundling

The manager at this factory has to keep track of how various shipments are bundled. Here's the system that is used.

The shipment of 46 objects to Pittsfield, Massachusetts is represented like this:

CARTONS	BOXES	PACKETS	OBJECTS
1	2	0	1

A shipment of 77 objects to Monterey, Massachusetts is represented like this:

CARTONS	BOXES	PACKETS	OBJECTS
2	2	1	2

place value

When you bundle in groups of three, you are using a base three number system. Notice that in a base three number system, the only digits used are 0, 1, and 2 . The digit 3 is never used, because whenever you have three of something, it gets bundled up into a bigger unit. The position labels (objects, packets, boxes, cartons) are called the *place values*.

A '2' in the 'object' place value represents 2 objects.

A '2' in the 'packet' place value represents 2 packets, hence $2 \cdot 3 = 6$ objects.

A '2' in the 'boxes' place value represents 2 boxes, hence $2 \cdot 9 = 18$ objects.

A '2' in the 'carton' place value represents 2 cartons, hence $2 \cdot 27 = 54$ objects.

EXERCISES

3. What is the largest shipment that can be represented using only carton, box, packet, and object place values? (Remember that the largest number you can have in any place value is 2 .)
4. Suppose that a new place value is introduced: three cartons get bundled into a *trunk*. What is the largest shipment that can be represented using trunk, carton, box, packet, and object place values?

exponent notation

Often, a good notation can help to clarify an idea. Next we will briefly discuss *exponent notation*, which will help to clarify the structure of the place values. Exponent notation will be discussed in more detail in a future section.

The number $7 \cdot 7$ is represented in exponent notation as 7^2 (read as '7 squared'). Notice that the exponent tells how many times 7 appears in the multiplication problem.

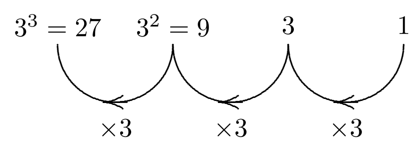
The number $10 \cdot 10 \cdot 10 \cdot 10$ is represented in exponent notation as 10^4 (read as '10 to the fourth power'). Notice again that the exponent tells how many times 10 appears in the multiplication problem.

EXERCISES

5. Write $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$ in exponent notation.
6. Write 2^5 without exponent notation.

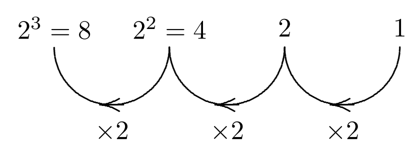
exponent notation for the place values in base three

In the base three number system, the place values are as follows:

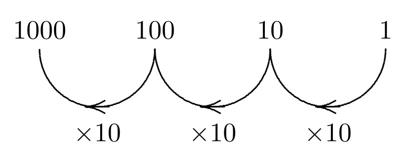


Notice that the right-most place value is 1 . Then, you multiply by 3 each time you move to the left.

In a base two number system, the place values are:



In our base ten number system, the place values are:



EXERCISES

7. From right to left, what are the first six place values in a base four number system?

base ten number system; names for place values

In our base ten number system, the place values are given the following names:

BILLION	HUNDRED MILLION	TEN MILLION	MILLION	HUNDRED THOUSAND	TEN THOUSAND	THOUSAND	HUNDRED	TEN	ONE
10^9	10^8	10^7	10^6	10^5	10^4	10^3	10^2	10	1

Therefore, for example, the number 23,481 represents:

$$\begin{aligned} 23{,}481 &= 2 \cdot 10^4 + 3 \cdot 10^3 + 4 \cdot 10^2 + 8 \cdot 10 + 1 \\ &= 2 \cdot 10{,}000 + 3 \cdot 1{,}000 + 4 \cdot 100 + 8 \cdot 10 + 1 \\ &= 2 \text{ ten-thousands} + 3 \text{ thousands} + 4 \text{ hundreds} + 8 \text{ tens} + 1 \text{ one} \end{aligned}$$

alternate groupings

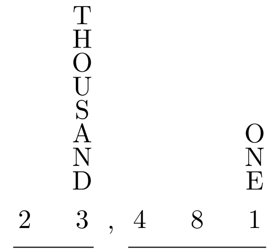
It is sometimes convenient to interpret numbers in a slightly different way. Again consider the number 23,481 , but suppose for the moment that you're only allowed to make piles of ten-thousands, piles of hundreds, and piles of ones. Then, you'd have the following:

TEN THOUSAND	HUNDRED	ONE
2	3 , 4	8 1

$$23{,}481 = 2 \text{ ten-thousands} + 34 \text{ hundreds} + 81 \text{ ones} .$$

Notice that 34 hundreds is the same as 3 thousands plus 4 hundreds, and 81 ones is the same as 8 tens plus 1 one.

If you're only allowed to make piles of thousands and piles of ones, then you'd have:



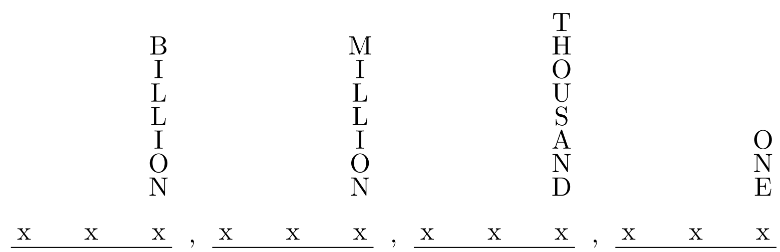
$$23{,}481 = 23 \text{ thousands} + 481 \text{ ones}\,.$$

This last re-naming corresponds to the scheme we use for reading numbers aloud: 23,481 is read as 'twenty-three thousand, four hundred eighty-one'.

reading numbers aloud;

resist the temptation to insert the word 'and'

Only the right-most place value in each group of three is used when we read numbers aloud in our base ten number system:



For example, $24,501,392,007$ is read aloud as: 'twenty-four billion, five hundred one million, three hundred ninety-two thousand, seven'. Resist the temptation to insert the word 'and' when you're reading numbers aloud, since you'll see in the next section that the word 'and' is reserved for the decimal point.

EXERCISES

8. Write the number $237,508$ in each of the following ways:

a. _____ hundred-thousands
 + _____ ten-thousands
 + _____ thousands
 + _____ hundreds
 + _____ tens
 + _____ ones

b. _____ hundred-thousands
 + _____ thousands
 + _____ hundreds
 + _____ ones

c. _____ ten-thousands
 + _____ hundreds
 + _____ ones

d. _____ hundred-thousands
 + _____ ten-thousands
 + _____ tens
 + _____ ones

EXERCISES

9. State how you would read aloud each of the following numbers:
   a. 43,509
   b. 14,473,820
   c. 43,037,008
   d. $3{,}000, 025{,}071$

multiplying by ten

In the base ten number system, it is extremely easy to multiply by powers of ten.

To multiply by $10^1 = 10$, put 1 zero at the end of the number: $237 \cdot 10 = 2{,}370$.

To multiply by $10^2 = 100$, put 2 zeros at the end of the number: $237 \cdot 100 = 23{,}700$.

To multiply by 10^n (which is 1 followed by n zeroes), put n zeros at the end of the number. For example, $237 \cdot 10^7 = 2{,}370{,}000{,}000$.

Think about why this is so easy! When, say, 237 is multiplied by 10, the 2 hundreds become 2 thousands; the 3 tens become 3 hundreds; and the 7 ones become 7 tens. Each digit needs to shift into the next-left place value. Putting the zero at the end of the number accomplishes this.

EXERCISES

10. Mentally compute the following multiplications:
    a. $23 \cdot 10^4$
    b. $438 \cdot 1{,}000$
    c. $5 \cdot 10^3$
    d. $1, 029 \cdot 100$

dividing by ten

It is equally easy to divide by powers of ten, but this requires knowledge of the place values to the right of the ones place—i.e., decimals. Decimals are the subject of the next section.

EXERCISES
web practice

Go to my homepage **https://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTIONS TO EXERCISES: THE BASE TEN NUMBER SYSTEM

1. $31 = 1\cdot 27 + 1\cdot 3 + 1\cdot 1$; one carton, one packet, and one object.

2. $68 = 2\cdot 27 + 1\cdot 9 + 1\cdot 3 + 2\cdot 1$; two cartons, one box, one packet, and two objects.

3. The largest shipment that can be represented is two cartons, two boxes, two packets, and two objects: $2\cdot 27 + 2\cdot 9 + 2\cdot 3 + 2\cdot 1 = 80$. Thus, the largest shipment that can be represented is 80 .

4. Notice that a trunk holds three cartons, or $3\cdot 27 = 81$ objects.

The largest shipment that can be represented is two trunks, two cartons, two boxes, two packets, and two objects: $2\cdot 81 + 2\cdot 27 + 2\cdot 9 + 2\cdot 3 + 2\cdot 1 = 242$. Thus, the largest shipment that can be represented is 242 .

5. $4\cdot 4\cdot 4\cdot 4\cdot 4\cdot 4 = 4^6$

6. $2^5 = 2\cdot 2\cdot 2\cdot 2\cdot 2 = 32$

7. From right to left: 1, 4, $4^2 = 16$, $4^3 = 64$, $4^4 = 256$, and $4^5 = 1024$

8.

a. 237,508 = 2 hundred-thousands
+ 3 ten-thousands
+ 7 thousands
+ 5 hundreds
+ 0 tens
+ 8 ones

b. 237,508 = 2 hundred-thousands
+ 37 thousands
+ 5 hundreds
+ 8 ones

c. 237,508 = 23 ten-thousands
+ 75 hundreds
+ 8 ones

d. 237,508 = 2 hundred-thousands
+ 3 ten-thousands
+ 750 tens
+ 8 ones

9. a. 43,509 : forty-three thousand, five hundred nine

b. 14,473,820 : fourteen million, four hundred seventy-three thousand, eight hundred twenty

c. 43,037,008 : forty-three million, thirty-seven thousand, eight

d. 3,000, 025,071 : three billion, twenty-five thousand, seventy-one

10. a. $23 \cdot 10^4 = 230{,}000$

b. $438 \cdot 1{,}000 = 438{,}000$

c. $5 \cdot 10^3 = 5{,}000$

d. $1,029 \cdot 100 = 102{,}900$