37. FINISHING UP ABSOLUTE VALUE INEQUALITIES

solving inequalities involving absolute value

This section should feel remarkably similar to the previous two. The only type of absolute value inequality that remains to be investigated is the one involving 'greater than,' like these:

$$|x| > 5$$
$$|x| \ge 3$$
$$|2 - 3x| \ge 7$$

Each of these inequalities has only a single set of absolute value symbols which is by itself on the left-hand side of the sentence, and has a *variable* inside the absolute value. The verb is either '$>$' or '$\ge$'. As in the previous section, solving sentences like these is easy, if you remember the critical fact that

absolute value gives distance from 0.

Keep this in mind as you read the following theorem:

THEOREM

tool for solving absolute value inequalities involving 'greater than'

Let $x \in \mathbb{R}$, and let $k > 0$. Then,

$$|x| > k \iff (x > k \text{ or } x < -k)$$
$$|x| \ge k \iff (x \ge k \text{ or } x \le -k)$$

$|x| > k$ is an entire class of sentences

Recall first that normal mathematical conventions dictate that '$|x| > k$' represents an entire class of sentences, including $|x| > 2$, $|x| > 5.7$, and $|x| > \frac{1}{3}$. The variable k changes from sentence to sentence, but is constant within a given sentence.

EXERCISES

1. a. Give three sentences of the form '$|x| > k$' where $k > 0$. Use examples different from those given above.
   b. Give three sentences of the form '$(x > k \text{ or } x < -k)$' where $k > 0$.

reminder: the mathematical word 'or'

Recall that an 'or' sentence is true when at least one of the subsentences is true. That is, 'A or B' is true when A is true, or when B is true, or when both A and B are true.

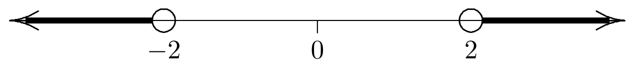
With this in mind, consider the values of x for which the compound sentence

$$\overbrace{x > 2}^{A} \quad \text{or} \quad \overbrace{x < -2}^{B}$$

is true:

- The sentence is true for any value of x that makes A true; i.e., for values of x greater than 2.
- The sentence is true for any value of x that makes B true; i.e., for values of x less than -2.
- There are no values of x for which A and B are both true at the same time.

Thus, the numbers that make '$x > 2$ or $x < -2$' true are shaded below:



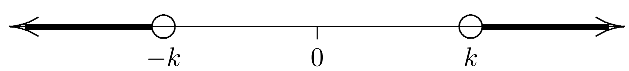
translating the theorem: thought process for solving sentences like $|x| > k$

When you see a sentence of the form '$|x| > k$', here's what you should do:

- Check that k is a positive number.
- The symbol $|x|$ represents the distance between x and 0.
- Thus, you want the numbers x, whose distance from 0 is greater than k.

$$\overbrace{|x|}^{\text{want numbers } x\text{, whose distance from 0}} \qquad \overbrace{>}^{\text{is greater than}} \qquad \overbrace{k}^{k}$$

- You can walk from 0 in two directions: more than k units to the right, or more than k units to the left. So, you want all the numbers to the right of k, together with all the numbers to the left of $-k$.



- Thus, $|x| > k$ is equivalent to $x > k$ or $x < -k$.

filling in some blanks to help your thought process

When you see a sentence like '$|x| > 7$', you thought process should be like filling in the following blanks:

We want the numbers _____ , whose _____ from _____ is greater than _____ . Thus, we want _____ to be greater than _____ or less than _____ .

The correctly-filled-in blanks are:

We want the numbers $\underline{x}$, whose <u>distance</u> from $\underline{0}$ is greater than $\underline{7}$. Thus, we want $\underline{x}$ to be greater than $\underline{7}$ or less than $\underline{-7}$.

EXERCISES

2. Fill in the blanks:
   a. When you look at the sentence ' $|x| > 5$ ', you should think: We want the numbers _____ , whose _____ from _____ is greater than _____ . Thus, we want _____ to be greater than _____ or less than _____ .
   b. When you look at the sentence ' $|z| > \frac{1}{5}$ ', you should think: We want the numbers _____ , whose _____ from _____ is greater than _____ . Thus, we want _____ to be greater than _____ or less than _____ .
   c. When you look at the sentence ' $|x| > k$ ' (with $k > 0$), you should think: We want the numbers _____ , whose _____ from _____ is greater than _____ . Thus, we want _____ to be greater than _____ or less than _____ .
3. Give a sentence, not using absolute value symbols, that is equivalent to:
   a. $|x| > 3$
   b. $|t| \ge 4.2$
4. Give a sentence, using absolute value symbols, that is equivalent to:
   a. $x > 7$ or $x < -7$
   b. $t \ge \frac{1}{3}$ or $t \le -\frac{1}{3}$
5. Give a precise mathematical statement of the tool that says that a sentence like ' $|x| > 5$ ' can be transformed to the equivalent sentence ' $x > 5$ or $x < -5$ '.
6. Is the sentence ' $|x| > -6$ ' of the form described in the previous theorem? Why or why not?
7. Can the sentence ' $|x| - 5 > 7$ ' be transformed to a sentence of the form described in the previous theorem? If so, what is the equivalent sentence?

EXAMPLE

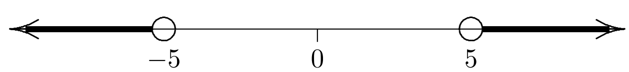
solving a sentence of the form $|x| > k$

Example: Solve: $|x| > 5$

Solution:

$$|x| > 5$$
$$x > 5 \quad \text{or} \quad x < -5$$

The solution set is shaded below:



solving more complicated sentences of the form $|x| > k$;

x can be ANYTHING!

The power of the tool

$$|x| > k \iff x > k \text{ or } x < -k$$

goes way beyond solving simple sentences like ' $|x| > 5$ '. Since x can be *any* real number, you should think of x as merely representing *the stuff inside the absolute value symbols*. Thus, you could think of rewriting the tool as:

$$|\text{stuff}| > k \iff \text{stuff} > k \text{ or } \text{stuff} < -k$$

Thus, we have all the following equivalences:

$$|\overbrace{2-3x}^{\text{stuff}}| > 7 \iff \overbrace{2-3x}^{\text{stuff}} > 7 \text{ or } \overbrace{2-3x}^{\text{stuff}} < -7$$

$$|\overbrace{5x-1}^{\text{stuff}}| > 8 \iff \overbrace{5x-1}^{\text{stuff}} > 8 \text{ or } \overbrace{5x-1}^{\text{stuff}} < -8$$

$$|\overbrace{x^2-3x+4}^{\text{stuff}}| > \frac{1}{5} \iff \overbrace{x^2-3x+4}^{\text{stuff}} > \frac{1}{5} \text{ or } \overbrace{x^2-3x+4}^{\text{stuff}} < -\frac{1}{5}$$

and so on!

EXERCISES

8. For each of the following, write an equivalent sentence that does not use absolute value symbols. Do *not* solve the resulting sentences.
   a. $|1+2x| > 3$
   b. $|7x - \frac{1}{2}| \ge 5$
   c. $|x^2 - 8| > 0.4$

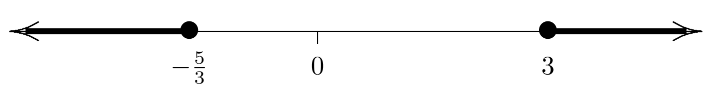
EXAMPLE

Example: Solve: $|2-3x| \ge 7$

Solution: Be sure to write a nice, clean list of equivalent sentences.

$\|2-3x\| \ge 7$	original sentence
$2-3x \ge 7$ or $2-3x \le -7$	$\|x\| \ge k \iff x \ge k$ or $x \le -k$
$-3x \ge 5$ or $-3x \le -9$	subtract 2 from both sides
$x \le -\frac{5}{3}$ or $x \ge 3$	divide both sides by -3

The solution set is shaded below:



Checking the boundary values:

$$|2 - 3(-\frac{5}{3})| \overset{?}{\geq} 7$$
$$|2 + 5| \overset{?}{\geq} 7$$
$$7 \geq 7$$

$$|2 - 3(3)| \overset{?}{\geq} 7$$
$$|-7| \overset{?}{\geq} 7$$
$$7 \geq 7$$

Checking a value of x for which the sentence should be false; choose $x = 0$:

$$|2 - 3(0)| \overset{?}{\geq} 7$$
$$2 \geq 7 \text{ is false!}$$

EXERCISES

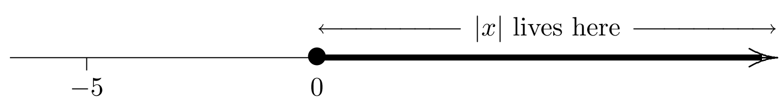
9. Solve. Write a nice, clean list of equivalent sentences. Spot-check and/or check boundary values on your solutions.
   a. $|1 + 2x| > 3$
   b. $|7x - 5| \geq 2$
   c. $|4 - 9x| > 1$

What happens if k is negative in the sentence '$|x| > k$'?

What about a sentence like '$|x| > -5$', where the absolute value is greater than a negative number? Notice that this situation is not covered in the previous theorem, since the sentence '$|x| > k$' is only addressed with $k > 0$.

Recall that $|x| \geq 0$ for all real numbers x. Thus, the sentence '$|x| > -5$' is *always* true! The left-hand side is a nonnegative number, which is guaranteed to be greater than -5.

$$\overbrace{|x|}^{\geq 0} > \overbrace{-5}^{<0}$$



Here are a few values of x substituted into '$|x| > -5$', to illustrate what is happening:

x	substitution into '$\|x\| > -5$'	simplifying sentence	true or false?
5	$\|5\| > -5$	$5 > -5$	true
-5	$\|-5\| > -5$	$5 > -5$	true
0	$\|0\| > -5$	$0 > -5$	true
2	$\|2\| > -5$	$2 > -5$	true

Whenever you see a sentence of the form '$|x| >$ *a negative number*', then you should **STOP** and say that the solution set is $\mathbb{R}$ (the set of all real numbers).

first step when analyzing $|x| > k$: check that $k > 0$

Whenever you're working with a sentence of the form '$|x| > k$', you must always check first that $k > 0$. If k is negative, you just stop and say that the sentence is always true. Here are some examples, which illustrate different ways that you can state your answer:

'$|x| > -3$' is always true.

'$|2x - 1| > -5$' is true for all real numbers x.

'$|3 - 4x| > -\frac{1}{7}$' is true for all $x \in \mathbb{R}$.

'$|3x - 5x^2 + 7| > -0.4$' has solution set $\mathbb{R}$.

EXERCISES

10. Decide which of the following sentences are always true. If it is sometimes true/sometimes false, so state.
    a. $|x| > -9$
    b. $|x| \ge 0$
    c. $|3x - 5| > -4.7$
    d. $|1 - 4x| + 5 \ge 0$
    e. $-2|x^2 + 3x - 1| < 8$
    f. $|9x + 1| - 5 \ge -3$

EXAMPLE

putting a sentence in standard form first

Sometimes you need a few transformations to get an equivalent sentence in the form $|x| > k$, as the next example illustrates. Remember that your goal is always to *isolate* the absolute value; i.e., get it all by itself on one side of the sentence (usually the left-hand side).

Solve: $5 - 2|3 - 4x| \le -7$

$5 - 2\|3 - 4x\| \le -7$	original sentence
$-2\|3 - 4x\| \le -12$	subtract 5 from both sides
$\|3 - 4x\| \ge 6$	divide both sides by -2; change verb
$3 - 4x \ge 6$ or $3 - 4x \le -6$	$\|x\| \ge k \iff (x \ge k \text{ or } x \le -k)$
$-4x \ge 3$ or $-4x \le -9$	subtract 3
$x \le -\dfrac{3}{4}$ or $x \ge \dfrac{9}{4}$	divide by -4; change verbs

A spot-check is left to you.

EXERCISES

11. Solve and check each of the following sentences. Be sure to write a nice, clean list of equivalent sentences.
    a. $7 - 5|1 - 2x| < -3$
    b. $-3|2x - 1| - 5 \le -4$
    c. $2|3x - 5| - 1 \ge 7$

EXERCISES

web practice

Go to my homepage `https://onemathematicalcat.org` and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTION TO EXERCISES: FINISHING UP ABSOLUTE VALUE INEQUALITIES

1. a. $|x| > 1$, $|x| > 7.2$, and $|x| > \frac{1}{2}$

b. $x > 1$ or $x < -1$, $x > 7.2$ or $x < -7.2$, and $x > \frac{1}{2}$ or $x < -\frac{1}{2}$

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2. a. We want the numbers $\underline{x}$, whose distance from $\underline{0}$ is greater than $\underline{5}$. Thus, we want $\underline{x}$ to be greater than $\underline{5}$ or less than $\underline{-5}$.

b. We want the numbers $\underline{z}$, whose distance from $\underline{0}$ is greater than $\underline{\frac{1}{5}}$. Thus,we want $\underline{z}$ to be greater than $\underline{\frac{1}{5}}$ or less than $\underline{-\frac{1}{5}}$.

c. We want the numbers $\underline{x}$, whose distance from $\underline{0}$ is greater than $\underline{k}$. Thus, we want $\underline{x}$ to be greater than $\underline{k}$ or less than $\underline{-k}$.

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3. a. $|x| > 3$ is equivalent to $x > 3$ or $x < -3$

b. $|t| \ge 4.2$ is equivalent to $x \ge 4.2$ or $x \le -4.2$

4. a. $x > 7$ or $x < -7$ is equivalent to $|x| > 7$

b. $t \ge \frac{1}{3}$ or $t \le -\frac{1}{3}$ is equivalent to $|t| \ge \frac{1}{3}$

5. For all real numbers x , and for $k > 0$, $|x| > k$ is equivalent to $x > k$ or $x < -k$.

6. The sentence '$|x| > -6$' is not of the form in the theorem, because -6 is a negative number.

7. The sentence '$|x| - 5 > 7$' can be transformed to '$|x| > 12$' by adding 5 to both sides.

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8. a. $|1 + 2x| > 3$ is equivalent to $1 + 2x > 3$ or $1 + 2x < -3$

b. $|7x - \frac{1}{2}| \ge 5$ is equivalent to $7x - \frac{1}{2} \ge 5$ or $7x - \frac{1}{2} \le -5$

c. $|x^2 - 8| > 0.4$ is equivalent to $x^2 - 8 > 0.4$ or $x^2 - 8 < -0.4$

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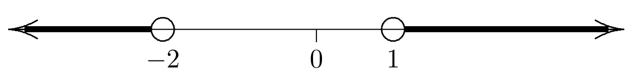
9. a.

$$\begin{gathered} |1 + 2x| > 3 \\ 1 + 2x > 3 \ \text{ or } \ 1 + 2x < -3 \\ 2x > 2 \ \text{ or } \ 2x < -4 \\ x > 1 \ \text{ or } \ x < -2 \end{gathered}$$

The solution set is:



Spot-check:

Choose a value of x greater than 1 ; choose (say) $x = 2$:

$$\begin{gathered} |1 + 2(2)| \overset{?}{>} 3 \\ 5 > 3 \ \text{ is true} \end{gathered}$$

Choose a value of x between -2 and 1 ; choose (say) $x = 0$:

$$\begin{gathered} |1 + 2(0)| \overset{?}{>} 3 \\ 1 > 3 \ \text{ is false} \end{gathered}$$

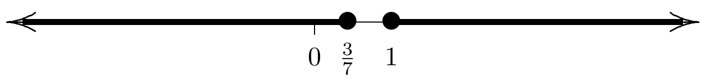
Choose a value of x less than -2; choose (say) $x = -3$:

$$|1 + 2(-3)| \overset{?}{>} 3$$
$$5 > 3 \text{ is true}$$

b.

$$|7x - 5| \geq 2$$
$$7x - 5 \geq 2 \text{ or } 7x - 5 \leq -2$$
$$7x \geq 7 \text{ or } 7x \leq 3$$
$$x \geq 1 \text{ or } x \leq \frac{3}{7}$$

The solution set is:



Spot-check:
Choose a value of x greater than 1; choose (say) $x = 2$:

$$|7(2) - 5| \overset{?}{\geq} 2$$
$$9 \geq 2 \text{ is true}$$

Choose a value of x between $\frac{3}{7}$ and 1; choose (say) $x = \frac{1}{2}$:

$$|7(\frac{1}{2}) - 5| \overset{?}{\geq} 2$$
$$1.5 \geq 2 \text{ is false}$$

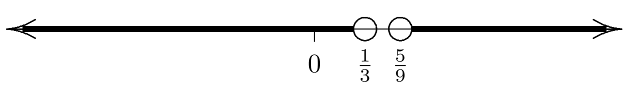
Choose a value of x less than $\frac{3}{7}$; choose (say) $x = 0$:

$$|7(0) - 5)| \overset{?}{\geq} 2$$
$$5 \geq 2 \text{ is true}$$

c.

$$|4 - 9x| > 1$$
$$4 - 9x > 1 \text{ or } 4 - 9x < -1$$
$$-9x > -3 \text{ or } -9x < -5$$
$$x < \frac{1}{3} \text{ or } x > \frac{5}{9}$$

The solution set is:



Spot-check:
Choose a value of x greater than $\frac{5}{9}$; choose (say) $x = 1$:

$$|4 - 9(1)| \overset{?}{>} 1$$
$$5 > 1 \text{ is true}$$

Choose a value of x between $\frac{1}{3} = \frac{3}{9}$ and $\frac{5}{9}$; choose (say) $x = \frac{4}{9}$:

$$|4 - 9(\frac{4}{9})| \overset{?}{>} 1$$
$$0 > 1 \text{ is false}$$

Choose a value of x less than $\frac{1}{3}$; choose (say) $x = 0$:

$$|4 - 9(0)| \overset{?}{>} 1$$
$$4 > 1 \text{ is true}$$

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10. (a) $|x| > -9$ is always true.

(b) $|x| \geq 0$ is true for all real numbers x.

(c) $|3x - 5| > -4.7$ is true for all $x \in \mathbb{R}$.

(d) $|1 - 4x| + 5 \geq 0$ is equivalent to $|1 - 4x| \geq -5$, which has solution set $\mathbb{R}$.

(e) $-2|x^2 + 3x - 1| < 8$ is equivalent to $|x^2 + 3x - 1| > -4$, which is always true.

(f) $|9x + 1| - 5 \geq -3$ is equivalent to $|9x + 1| \geq 2$; it is sometimes true, sometimes false.

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11. a.

$$7 - 5|1 - 2x| < -3$$
$$-5|1 - 2x| < -10$$
$$|1 - 2x| > 2$$
$$1 - 2x > 2 \text{ or } 1 - 2x < -2$$
$$-2x > 1 \text{ or } -2x < -3$$
$$x < -\frac{1}{2} \text{ or } x > \frac{3}{2}$$

Spot-checks: (A compact way of writing down the spot-checks is illustrated here.)

$x = -1$ (should be true): $7 - 5|1 - 2(-1)| = 7 - 5|1 + 2| = 7 - 5(3) = 7 - 15 = -8$ is less than -3

$x = 0$ (should be false): $7 - 5|1 - 2(0)| = 7 - 5|1| = 7 - 5 = 2$ is not less than -3

$x = 2$ (should be true): $7 - 5|1 - 2(2)| = 7 - 5|-3| = 7 - 5(3) = 7 - 15 = -8$ is less than -3

b.

$$-3|2x - 1| - 5 \leq -4$$
$$-3|2x - 1| \leq 1$$
$$|2x - 1| \geq -\frac{1}{3}$$
$$\text{always true!}$$

c.

$$2|3x - 5| - 1 \geq 7$$
$$2|3x - 5| \geq 8$$
$$|3x - 5| \geq 4$$
$$3x - 5 \geq 4 \text{ or } 3x - 5 \leq -4$$
$$3x \geq 9 \text{ or } 3x \leq 1$$
$$x \geq 3 \text{ or } x \leq \frac{1}{3}$$

Spot-checks:

$x = 0$ (should be true): $2|3(0) - 5| - 1 = 2|-5| - 1 = 2(5) - 1 = 10 - 1 = 9$ is greater than or equal to 7.

$x = 1$ (should be false): $2|3(1) - 5| - 1 = 2|-2| - 1 = 2(2) - 1 = 4 - 1 = 3$ is not greater than or equal to 7.

$x = 4$ (should be true): $2|3(4) - 5| - 1 = 2|7| - 1 = 2(7) - 1 = 14 - 1 = 13$ is greater than or equal to 7.

