26. FOIL AND MORE

extending the distributive law

In this section, the distributive law is extended to cover situations like these:

$$a(b+c+d) = ab+ac+ad$$
$$(a+b)(c+d) = ac+ad+bc+bd$$
$$(a+b+c)(d+e+f) = ad+ae+af+bd+be+bf+cd+ce+cf$$

Along the way, an important memory device called FOIL (**F**irst **O**uter **I**nner **L**ast) is presented.

First, some new terminology is needed:

terms;

In an addition problem (a *sum*), the things being added are called the *terms*.

identifying terms

For example, in the addition problem $2+3x+5y$, the terms are 2, $3x$, and $5y$.

a term includes its sign

If subtraction is involved, rewrite the expression (at least mentally) as an addition problem before identifying the terms. Here's an example: to find the terms in the expression $1-2t+3xy-5x^2$, first rewrite as follows:

$$1-2t+3xy-5x^2 = 1+(-2t)+3xy+(-5x^2)$$

The terms are 1, $-2t$, $3xy$, and $-5x^2$. Notice that a minus sign becomes 'part of' a term. The phrase used to describe this situation is that *a term includes its sign*.

numerical coefficient; coefficient

In a term like $2x$, there are two parts that are usually of interest: the numerical part (2) and the variable part (x). The numerical part is given a special name—it is called the *numerical coefficient* or, more simply, the *coefficient* of the term.

In the term $4xy$, the coefficient is 4 and the variable part is xy.

In the term $-7x^2y^3$, the coefficient is -7 and the variable part is x^2y^3.

What if you don't 'see' a coefficient?

If you don't 'see' a coefficient, then it is 1. That is, $x=(1)x$ has coefficient 1. Also, $x^2y^3=(1)x^2y^3$ has coefficient 1. It's never necessary to write a coefficient of 1, because multiplication by 1 doesn't change anything.

What is the coefficient in the term $-x$?

In the term $-x$, the coefficient is -1, because $-x=(-1)x$.

In the term $-x^2y$, the coefficient is -1, because $-x^2y=(-1)x^2y$.

constant terms

A term like 2 that has no variable part is called a *constant term*, because it is constant—it never changes. It has no variable part that can 'hold' different values. Thus, $\frac{1}{2}$, $\sqrt{3}$, 9.4, $9\sqrt{3}$, and $-\frac{8}{103}$ are all constant terms.

convention for writing terms:

In any term, it is conventional to write the numerical coefficient first. Thus, you should write $4xy$, not $xy4$ or $x4y$.

coefficient first, variable part last, alphabetical order

Also, it is conventional to write any variable(s) in alphabetical order. Thus, you usually want to write $5xy$, not $5yx$. You should write x^2yz, not yx^2z or yzx^2.

EXERCISES	1. In each problem, list the terms. How many terms are there? State the variable part and coefficient of each term. Identify any constant term(s). a. $2y - 3xy + 4x^2$ b. $-5 + \sqrt{2}x - y + z$

like terms:

the same variable part

Terms with the same variable part are called *like terms* because they look 'alike' as far as the variable part is concerned. The phrase *like terms* can refer to two or more terms.

Thus, $2x$ and $\frac{1}{3}x$ are like terms. In each term, the variable part is x.

Also, x^2, $\frac{1}{3}x^2$, and $\sqrt{5}x^2$ are like terms. In each term, the variable part is x^2.

combining like terms

Only like terms can be combined, and they are combined by adding the coefficients. For example, $2x + 5x = 7x$ and $7y - 4y = 3y$.

Terms that are *not* like terms can *not* be combined. For example, there is no simpler way to write $2x + 5y$ or $7y - 2y^2$.

Why can like terms be combined?

The reason that like terms can be combined is a consequence of the distributive law, in 'backwards' form! Keep reading!

Recall the distributive law: $a(b + c) = ab + ac$.

Writing it from right to left, we have: $ab + ac = a(b + c)$.

Consider the pattern '$ab + ac$': the terms are ab and ac. The term ab has a factor 'a'. The term ac has a factor 'a'. The factor 'a' is common to both terms.

Notice that if you 'remove' the common part, a, from the term ab, the part left is b.

Notice that if you 'remove' the common part, a, from the term ac, the part left is c.

In rewriting $ab + ac$ as $a(b + c)$, the common factor a is written down first, and then an opening parenthesis is inserted. Then, the part remaining from each term is written down:

$$\underline{a}b + \underline{a}c = \underline{a}(b + c)$$

Here's how it looks in a different situation:

$$2x + 3x = 2\underline{x} + 3\underline{x} = \underline{x}(2 + 3) = x(5) = 5x\,.$$

Thus, the distributive law is the reason that $2x + 3x = 5x$.

See how important the distributive law is! The ideas discussed in this paragraph will be explored in more detail in future sections on factoring.

EXAMPLES

combining like terms

Here are examples of combining like terms:

Example:

$2x - y - 5x + 6y = -3x + 5y$

Note that $2x - 5x = -3x$ and $-y + 6y = 5y$. You want to be able to do a problem like this in one step.

Example:

$1 - x^2 + 2x - 3 + 7x^2 - 5x = 6x^2 - 3x - 2$

Note that $1 - 3 = -2$, $2x - 5x = -3x$, and $-x^2 + 7x^2 = 6x^2$.

In a sum that involves different powers of x, it is conventional to order the terms from the highest power down to the lowest power, with the constant term coming last.

Example:

$\frac{1}{2}x - \frac{2}{3}x = \frac{3}{6}x - \frac{4}{6}x = -\frac{1}{6}x$

Note the usual arithmetic with fractions.

Example:

$0.4y - 0.7y = -0.3y$

Note the usual arithmetic with decimals.

EXERCISES

2. Combine like terms. Write each result in the most conventional way. Be sure to write a complete mathematical sentence.
   a. $6x - 3y + 4x - 7y$
   b. $x - 1 + x^2 - 2x + 5 - 3x^2$
   c. $\frac{1}{3}y - \frac{2}{5}y$
   d. $1.4w + 0.2w - w$

There have been many definitions introduced over the last couple pages, which are summarized for completeness next:

DEFINITIONS

sum; terms; like terms; numerical coefficient; constant term

A *sum* is an addition problem.

In a sum, the things being added are called the *terms*.

Like terms are terms that have the same variable part.

A term with no variable part is called a *constant term*.

When writing terms, put any constant(s) before any variable(s), and write variables in alphabetical order.

The constant part of a term is called the *numerical coefficient* or, more simply, the *coefficient*.

$a(b+c+d)$
$= ab+ac+ad$
and more ...

Now we're ready to look at several extensions of the distributive law. Recall that the 'basic model' of the distributive law is:

$$a(b+c) = ab + ac$$

There may be more than two terms in the parentheses:

$$a(b+c+d) = ab + ac + ad$$
$$a(b+c+d+e) = ab + ac + ad + ae$$
$$\text{and so on}\ldots$$

All the usual rules for dealing with signed terms hold. For example,

$$-a(2b - c + 4d + f) = -2ab + ac - 4ad - af$$

Remember to determine the sign (plus or minus) first, the numerical part next, and the variable part last. Write all variables in alphabetical order within a given term.

Of course, you might see the group coming first, like this:

$$(2a - b + 3c + d)(-f) = -2af + bf - 3cf - df$$

EXERCISES

3. Use the distributive law. Write in the most conventional way. Be sure to write a complete mathematical sentence.
   a. $a(b+c+d+e+f)$
   b. $-a(4b - 3c + 2e)$
   c. $(-a + 5b - c)(-d)$
   d. $(2a - b + 3c)(-3d)$

the 'treat as a singleton' technique applied to $(a+b)(c+d)$

At first glance, it might not look like the distributive law applies to the expression $(a+b)(c+d)$. However, it does—once you apply a popular mathematical technique called 'treat it as a singleton'. Here's how it goes:

First, rewrite the distributive law using some different variable names:

$$z(c+d) = zc + zd\,.$$

This says that *anything* times $(c+d)$ is the *anything* times c, plus the *anything* times d.

Now, look back at $(a+b)(c+d)$, and take the group $(a+b)$ as z. That is, you're taking something that seems to have two parts, and you're treating it as a single thing, a 'singleton'! Look what happens:

$$
\begin{aligned}
(a+b)(c+d) &= \overbrace{(a+b)}^{\text{call this } z}(c+d) && \text{rename } (a+b) \text{ as } z\\
&= z(c+d) && \text{rewrite}\\
&= zc+zd && \text{use the distributive law}\\
&= (a+b)c+(a+b)d && z=a+b\\
&= ac+bc+ad+bd && \text{use the distributive law twice}\\
&= ac+ad+bc+bd && \text{re-order}\\
&= \overbrace{ac}^{\textbf{F}\text{irst}}+\overbrace{ad}^{\textbf{O}\text{uter}}+\overbrace{bc}^{\textbf{I}\text{nner}}+\overbrace{bd}^{\textbf{L}\text{ast}}
\end{aligned}
$$

FOIL:
F*irst*
O*uter*
I*nner*
L*ast*

Notice that $(a+b)(c+d) = \overbrace{ac}^{\textbf{F}\text{irst}}+\overbrace{ad}^{\textbf{O}\text{uter}}+\overbrace{bc}^{\textbf{I}\text{nner}}+\overbrace{bd}^{\textbf{L}\text{ast}}$. You get four terms, and each of these terms is assigned a letter. These letters form the word 'FOIL,' and provide a powerful memory device for multiplying out expressions of the form $(a+b)(c+d)$. Here's the meaning of each letter in the word 'FOIL':

- The first number in the group $(a+b)$ is a; the first number in the group $(c+d)$ is c. Multiplying these 'firsts' together gives ac, which is labeled '**F**irst':

$$(\overbrace{a}^{\textbf{F}\text{irst}}+b)(\overbrace{c}^{\textbf{F}\text{irst}}+d)$$

- When you look at the group $(a+b)(c+d)$ from far away, you see a and d on the outside. That is, a and d are the outer numbers. Multiplying these outer numbers together gives ad, which is labeled '**O**uter':

$$(\overbrace{a}^{\textbf{O}\text{uter}}+b)(c+\overbrace{d}^{\textbf{O}\text{uter}})$$

- Similarly, when you look at the group $(a+b)(c+d)$ from far away, you see b and c on the inside. That is, b and c are the inner numbers. Multiplying these inner numbers together gives bc, which is labeled '**I**nner':

$$(a+\overbrace{b)(c}^{\textbf{I}\text{nner}}+d)$$

- The last number in the group $(a+b)$ is b; the last number in the group $(c+d)$ is d. Multiplying these 'lasts' together gives bd, which is labeled '**L**ast':

$$(a+\overbrace{b}^{\textbf{L}\text{ast}})(c+\overbrace{d}^{\textbf{L}\text{ast}})$$

EXERCISES

4. a. Instead of FOIL, suppose someone uses 'FILO' as a memory device. List the corresponding order of the terms when expanding $(a+b)(c+d)$.

   b. Instead of FOIL, suppose someone uses 'LIFO' as a memory device. List the corresponding order of the terms when expanding $(a+b)(c+d)$.

   c. Why do you suppose that the common memory device has become 'FOIL,' and not (say) 'FILO' or 'LIFO'?

EXAMPLES
using FOIL

Here are some examples of using FOIL for multiplying expressions of the form $(a+b)(c+d)$. In all cases, you should be able to identify F, O, I, and L.

Example: $(e+f)(g+h) = \overbrace{eg}^{F} + \overbrace{eh}^{O} + \overbrace{fg}^{I} + \overbrace{fh}^{L}$

Notice in the following problems that you should use the same rules that have now been practiced extensively—figure out the sign (plus or minus) of each term first, the numerical part next, and the variable part last.

Example: $(2a+b)(c-d) = \overbrace{2ac}^{F} \overbrace{-\ 2ad}^{O} + \overbrace{bc}^{I} \overbrace{-\ bd}^{L}$

Notice that F is $2ac$; O is $-2ad$; I is bc; and L is $-bd$. Each term includes its sign.

Example: $(-3a+b)(c-4d) = \overbrace{-3ac}^{F} + \overbrace{12ad}^{O} + \overbrace{bc}^{I} \overbrace{-4bd}^{L}$

Example: $(-2a+3)(3b-1) = -6ab + 2a + 9b - 3$

EXERCISES

5. Use FOIL. In each case, identify F, O, I, and L.

   a. $(a-b)(c+d)$

   b. $(a-b)(c-d)$

   c. $(2a+b)(c-d)$

   d. $(2a-b)(3c-d)$

EXAMPLES
expressions of the form $(x+k_1)(x+k_2)$

One common application of FOIL is to multiply out expressions like $(x+3)(x+5)$ or $(x-1)(x+4)$. Remember the exponent laws, and be sure to combine like terms whenever possible:

Example: $(x-1)(x+3) = x^2 + 3x - x - 3 = x^2 + 2x - 3$

Example: $(x-2)(x-5) = x^2 - 5x - 2x + 10 = x^2 - 7x + 10$

Example: $(2x+1)(3x-4) = 6x^2 - 8x + 3x - 4 = 6x^2 - 5x - 4$

EXERCISES

6. Use FOIL. Be sure to combine like terms.
   a. $(x+2)(x+3)$
   b. $(x-2)(x+3)$
   c. $(x-2)(x-3)$
   d. $(x+2)(x-3)$
   e. $(2x+1)(x-3)$
   f. $(2x-1)(3x+4)$
   g. $(x-2)(3x-5)$

one more extension of the distributive law:

things like $(a+b)(c+d+e)$

To conclude this section, we look at one final extension of the distributive law. The 'treat it as a singleton' technique can be used (again!) to show that

$$(a+b)(c+d+e) = ac+ad+ae+bc+bd+be$$

Notice that there are 2 terms in the first group $(a+b)$ and 3 terms in the second group $(c+d+e)$. After expanding, there are $2 \cdot 3 = 6$ terms.

Notice also that the first member of the first group (a) multiplies each term in the second group. Then, the second member of the first group (b) multiplies each term in the second group:

$$(a+b)(c+d+e) = \overbrace{ac \ + \ ad \ + \ ae}^{\text{`}a\text{' multiples each term in second group}} \quad + \quad \overbrace{bc \ + \ bd \ + \ be}^{\text{`}b\text{' multiplies each term in second group}}$$

This pattern can be used whenever you have groups with two or more terms that are being multiplied.

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the 'treat it as a singleton' technique again

$$\begin{aligned}(a+b)(c+d+e) &= (a+b)(\overbrace{c+d}^{z}+e)\\ &= (a+b)(z+e)\\ &= az+ae+bz+be\\ &= a(c+d)+ae+b(c+d)+be\\ &= ac+ad+ae+bc+bd+be\end{aligned}$$

one final example

Here's another example, with an extra step written out:

$$\begin{aligned}&(a+b+c)(d+e+f+g)\\ &\qquad = a(d+e+f+g)+b(d+e+f+g)+c(d+e+f+g)\\ &\qquad = ad+ae+af+ag+bd+be+bf+bg+cd+ce+cf+cg\end{aligned}$$

Notice that there are 3 terms in the first group and 4 terms in the second group, and we ended up with $3 \cdot 4 = 12$ terms in the final expression.

EXERCISES

7. First compute how many terms there will be in the final result. Then, multiply out.
   a. $(a+b+c)(x+y)$
   b. $(a+b+c)(d+e+f)$
   c. $(a+b)(x+y+z+w)$

EXERCISES
web practice

Go to my homepage `https://onemathematicalcat.org` and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTION TO EXERCISES: FOIL AND MORE

1. a. $2y - 3xy + 4x^2$:
there are three terms: $2y$, $-3xy$, and $4x^2$
$2y$: variable part y , coefficient 2
$-3xy$: variable part xy , coefficient -3
$4x^2$: variable part x^2 , coefficient 4

b. $-5 + \sqrt{2}x - y + z$:
there are four terms: -5 , $\sqrt{2}x$, $-y$, and z
-5 : constant term
$\sqrt{2}x$: variable part x , coefficient $\sqrt{2}$
$-y$: variable part y , coefficient -1
z : variable part z , coefficient 1

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2. a. $6x - 3y + 4x - 7y = 10x - 10y$
b. $x - 1 + x^2 - 2x + 5 - 3x^2 = -2x^2 - x + 4$
c. $\frac{1}{3}y - \frac{2}{5}y = \frac{5}{15}y - \frac{6}{15}y = -\frac{1}{15}y$
d. $1.4w + 0.2w - w = (1.4 + 0.2 - 1)w = 0.6w$

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3. a. $a(b+c+d+e+f) = ab + ac + ad + ae + af$
b. $-a(4b - 3c + 2e) = -4ab + 3ac - 2ae$
c. $(-a + 5b - c)(-d) = ad - 5bd + cd$
d. $(2a - b + 3c)(-3d) = -6ad + 3bd - 9cd$

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4. a. FILO: $(a+b)(c+d) = ac + bc + bd + ad$
b. LIFO: $(a+b)(c+d) = bd + bc + ac + ad$
c. FOIL is an actual, common, word!

5. a. $(a-b)(c+d) = \overbrace{ac}^{F} + \overbrace{ad}^{O} \overbrace{- bc}^{I} \overbrace{- bd}^{L}$

b. $(a-b)(c-d) = \overbrace{ac}^{F} \overbrace{- ad}^{O} \overbrace{- bc}^{I} + \overbrace{bd}^{L}$

c. $(2a+b)(c-d) = \overbrace{2ac}^{F} \overbrace{- 2ad}^{O} + \overbrace{bc}^{I} \overbrace{- bd}^{L}$

d. $(2a-b)(3c-d) = \overbrace{6ac}^{F} \overbrace{- 2ad}^{O} \overbrace{- 3bc}^{I} + \overbrace{bd}^{L}$

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6. a. $(x+2)(x+3) = x^2 + 3x + 2x + 6 = x^2 + 5x + 6$

b. $(x-2)(x+3) = x^2 + 3x - 2x - 6 = x^2 + x - 6$

c. $(x-2)(x-3) = x^2 - 3x - 2x + 6 = x^2 - 5x + 6$

d. $(x+2)(x-3) = x^2 - 3x + 2x - 6 = x^2 - x - 6$

e. $(2x+1)(x-3) = 2x^2 - 6x + x - 3 = 2x^2 - 5x - 3$

f. $(2x-1)(3x+4) = 6x^2 + 8x - 3x - 4 = 6x^2 + 5x - 4$

g. $(x-2)(3x-5) = 3x^2 - 5x - 6x + 10 = 3x^2 - 11x + 10$

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7. a. $3 \cdot 2 = 6$ terms:

$(a+b+c)(x+y) = ax + ay + bx + by + cx + cy$

b. $3 \cdot 3 = 9$ terms:

$(a+b+c)(d+e+f) = ad + ae + af + bd + be + bf + cd + ce + cf$

c. $2 \cdot 4 = 8$ terms:

$(a+b)(x+y+z+w) = ax + ay + az + aw + bx + by + bz + bw$