20. EXPONENT NOTATION

you've already seen things like x^2 and x^3

Several times already in this text, you've been introduced to expressions like x^2 (a shorthand for $x \cdot x$) and x^3 (a shorthand for $x \cdot x \cdot x$). This type of notation is called *exponent notation*, and is pretty hard to avoid in mathematics because of its compactness and utility. This section will fill in some details about exponent notation; the next section will develop tools for working with expressions involving exponents.

We begin with a definition which summarizes the meaning of x^n for integer values of n. Rational values of n will be addressed in a future section. The definition is stated concisely and precisely, and then discussed in a more leisurely way in the paragraphs that follow.

DEFINITION

base; exponent; power

Let $x \in \mathbb{R}$. In the expression x^n, x is called the *base* and n is called the *exponent* or the *power*.

positive integers

If $n \in \{1, 2, 3, \dots\}$, then

$$x^n = \overbrace{x \cdot x \cdot x \cdot \ \dots \ \cdot x}^{n \text{ factors}}$$

In this case, x^n is just a shorthand for repeated multiplication.

Note that $x^1 = x$ for all real numbers x.

zero

If $x \neq 0$, then

$$x^0 = 1\,.$$

The expression 0^0 is not defined.

negative integers

If $n \in \{1, 2, 3, \dots\}$ and $x \neq 0$ then

$$x^{-n} = \frac{1}{x^n} = \frac{1}{\underbrace{x \cdot x \cdot x \cdot \ \dots \ \cdot x}_{n \text{ factors}}}\,.$$

In particular, $x^{-1} = \dfrac{1}{x^1} = \dfrac{1}{x}$ for all nonzero real numbers x.

That is, x^{-1} is the reciprocal of x.

MEMORY DEVICES *to help remember:*

Here are some memory devices that may help you remember several basic facts about exponents.

$x^0 = 1$

Turn an exponent of 0 into the word **O**ne, to remember that x to the 0 power is 1:

$$x^0 \ \text{ turns into } \ x^{\mathbf{0}\text{ne}} \quad \text{ to remember that } \quad x^0 = 1$$

$x^1 = x$

Turn an exponent of 1 into the word **I**tself, to remember that x to the 1 power is itself:

$$x^1 \ \text{ turns into } \ x^{\mathbf{1}\text{tself}} \quad \text{ to remember that } \quad x^1 = x$$

$x^{-1} = \frac{1}{x}$

Take an exponent of -1, slide the minus sign to the right, and turn it into the letter **R**, to remember that x to the -1 power is the reciprocal:

-1 —(slide over the minus sign)→ 1- —(turn it into the letter R)→ R

$$x^{-1} \ \text{ turns into } \ x^{\mathbf{R}\text{eciprocal}} \quad \text{ to remember that } \quad x^{-1} = \frac{1}{x}$$

EXERCISES

1. To test your skills in reading mathematics, try to answer the following questions *before* reading the rest of this section.
   a. What does the symbol $\mathbb{R}$ stand for?
   b. How should you read this aloud? ‘Let $x \in \mathbb{R}$.’
   c. In the expression y^m, what is y called? What is m called?
   d. In the expression 3^5, what is 3 called? What is 5 called?
   e. What are the integers?
   f. What are the positive integers? What are the negative integers?
   g. Is the sentence ‘$n \in \{1, 2, 3, \dots\}$’ true when $n = 7$? When $n = \frac{1}{2}$? When $n = -10$? When $n = \frac{200}{5}$?
   h. What is x^5 shorthand for?
   i. As long as $y \neq 0$, what is y^0?
   j. What is y^{-3} shorthand for? (assume $y \neq 0$)
   k. What is y^{-1} shorthand for? (assume $y \neq 0$)

The reason that zero and negative exponents are defined the way they are is discussed in the next section. For now, we'll practice with other parts of the definition.

positive integers n; repeated multiplication

When n is 1 or 2 or 3 and so on, then x^n is just a shorthand for repeated multiplication. Here are many examples, some of which are discussed in more detail in the following paragraphs.

expression	how to read aloud	shorthand for:
x^1	x to the first power	$x^1 = x$
x^2	x squared	$x^2 = x \cdot x$
x^3	x cubed	$x^3 = x \cdot x \cdot x$
x^4	x to the fourth power	$x^4 = x \cdot x \cdot x \cdot x$
x^5	x to the fifth power	$x^5 = x \cdot x \cdot x \cdot x \cdot x$
0^0	0 to the 0 power	undefined
0^{2007}	0 to the 2007^{th} power	$0^{2007} = 0$; 0 to any nonzero power is 0
1^{80}	1 to the 80^{th} power	$1^{80} = 1$; one to any power is 1
$(-1)^{80}$	negative one, to the 80^{th} power	$(-1)^{80} = 1$; -1 to any EVEN power is 1
1^{81}	1 to the 81^{st} power	$1^{81} = 1$; one to any power is 1
$(-1)^{81}$	negative one, to the 81^{st} power	$(-1)^{81} = -1$; -1 to any ODD power is -1
2^3	2 cubed	$2^3 = 2 \cdot 2 \cdot 2 = 8$
$(-2)^3$	negative two, cubed	$(-2)^3 = (-2)(-2)(-2) = -8$
-2^3	negative, two cubed	$-2^3 = (-1)(2^3) = (-1)(2 \cdot 2 \cdot 2) = -8$
2^4	2 to the fourth power	$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$
$(-2)^4$	negative two, to the fourth power	$(-2)^4 = (-2)(-2)(-2)(-2) = 16$
-2^4	negative, two to the fourth power	$-2^4 = (-1)(2^4) = (-1)(2 \cdot 2 \cdot 2 \cdot 2) = -16$

sign first; size second

When simplifying expressions involving exponent notation, figure out the *sign* (plus or minus) of the expression first, then figure out its *size*.

Recall that any even number $(2, 4, 6, \dots)$ of negative factors is positive.

Any odd number $(1, 3, 5, \dots)$ of negative factors is negative.

For example, consider $(-2)^6$. There are an even number (6) of negative factors, so the result is positive. The size of the result is $2^6 = 64$. Thus, $(-2)^6 = 64$.

As a second example, consider $(-2)^5$. There are an odd number (5) of negative factors, so the result is negative. The size of the result is $2^5 = 32$. Thus, $(-2)^5 = -32$.

$(-x)^{\text{even}} = x^{\text{even}}$

Notice that $(-2)^6 = 2^6$; both are positive, and both have the same size.

Also, $(-3)^{100} = 3^{100}$; both are positive, and both have the same size.

In general, $x^{\text{even}} = (-x)^{\text{even}}$ for all real numbers x . In words, when you raise a number and its opposite to the same even power, you get the same result.

EXERCISES

2. State the sign (plus or minus) of each expression.
   a. $(\text{positive})^{\text{even}}$
   b. $(\text{positive})^{\text{odd}}$
   c. $(\text{negative})^{\text{even}}$
   d. $(\text{negative})^{\text{odd}}$

3. State the sign (plus or minus) of each expression.
   a. x^7 , if x is negative
   b. x^7 , if x is positive
   c. $(-x)^7$, if x is negative
   d. $(-x)^7$, if x is positive
   e. $(-x)^8$, if x is negative
   f. $(-x)^8$, if x is positive

EXERCISES

4. Simplify each expression. Think about computing the sign first, and the size last.
   a. $(-3)^2$
   b. $(-3)^3$
   c. 1^{2008}
   d. $(-1)^{2008}$
   e. 1^{2009}
   f. $(-1)^{2009}$

order of operations; addition and multiplication are binary operations

We need to pause momentarily and talk about some order of operation issues.

Addition ($+$) is called a *binary operation* because it takes two inputs: given two numbers, they can be added.

Multiplication is also a *binary operation* because it also takes two inputs: given two numbers, they can be multiplied.

two operations competing for the same number

Now consider this expression (where $\times$ has been used to denote multiplication):

$$2 + 3 \times 4$$

There are two operations, and three numbers.

The ‘ $+$ ’ operation is trying to ‘grab’ the 2 and the 3 to add them.

The ‘ $\times$ ’ operation is trying to ‘grab’ the 3 and the 4 to multiply them.

Both operations are ‘competing’ for the number 3 .

Think of this as a sort of tug of war. Addition is pulling on 3 from the left. Multiplication is pulling on 3 from the right. Who is stronger? Who will win?

It would make a difference in the result. If addition wins, then we would get:

$$(2 + 3) \times 4 = 5 \times 4 = 20$$

However, if multiplication wins, then we would get:

$$2 + (3 \times 4) = 2 + 12 = 14$$

multiplication wins; multiplication is 'super-addition'

Mathematicians have decided that multiplication is the stronger operation, so multiplication 'wins'. This makes sense, since multiplication is like 'super-addition': for example, $4 \times 2 = 2 + 2 + 2 + 2$. Thus,

$$2 + 3 \times 4 = 2 + (3 \times 4) = 2 + 12 = 14$$

Notice that this is *not* a simple left-to-right simplifying process. Even though you come to the '$2 + 3$' first in traveling from left to right, it is *not* done first.

My Dear Aunt Sally

Multiplication and division are 'equally strong,' since division is a special type of multiplication.

Addition and subtraction are 'equally strong,' since subtraction is a special type of addition.

When addition, subtraction, multiplication, and division are all mixed up, do all multiplications and divisions in order, as they appear, going from left to right. Then, do all additions and subtractions in order, as they appear, going from left to right.

Some people remember the phrase '**M**y **D**ear **A**unt **S**ally' to remember that **M**ultiplication and **D**ivision get done before **A**ddition and **S**ubtraction.

For example,

$$\begin{aligned} 1 - 2 \times 3 + 10 \div 5 &= 1 - (2 \times 3) + (10 \div 5) \\ &= 1 - 6 + 2 \\ &= -3 \end{aligned}$$

EXERCISES

5. Simplify each expression.
   a. $1 + 3 \times 5 - 20 \div 4$
   b. $-2 + 10 \div 5 \times 3$
   c. $4 - 3 + 12 \div 6 \times 2 + 1$

   Check that your calculator gives the same results.

again: two operations competing for the same number

This time, let's consider what happens when multiplication is 'competing with' a power. Let $\wedge$ denote 'to the power of' in this discussion; so instead of writing $2 \cdot 3^4$, we'll write:

$$2 \times 3 \wedge 4$$

There are two operations, and three numbers.

The '$\times$' operation is trying to 'grab' the 2 and the 3 to multiply them.

The '$\wedge$' operation is trying to 'grab' the 3 and the 4 to raise 3 to the 4^{th} power.

Both operations are 'competing' for the number 3.

Another tug of war. Multiplication is pulling on 3 from the left. The power operation is pulling on 3 from the right. Who is stronger? Who will win?

It would make a difference in the result. If multiplication wins, then we would get:

$$(2 \times 3) \wedge 4 = 6 \wedge 4 = 1296$$

However, if the power wins, then we would get:

$$2 \times (3 \wedge 4) = 2 \times (3^4) = 2 \times 81 = 162$$

the power wins; a power is 'super-multiplication'

Mathematicians have decided that the power is the stronger operation, so it 'wins'. This makes sense, since a power is like 'super-multiplication': for example, $3^4 = 3 \times 3 \times 3 \times 3$. Thus,

$$2 \times 3 \wedge 4 = 2 \times (3 \wedge 4) = 2 \times (3^4) = 2 \cdot 81 = 162$$

Here's the way it is usually written:

$$2 \cdot 3^4 = 2 \cdot 81 = 162$$

Again, notice that this is *not* a simple left-to-right simplifying process. Even though you come to the 2×3 first in traveling from left to right, it is *not* done first.

Combining results thus far: powers are stronger than multiplications/divisions, which are stronger than additions/subtractions.

EXERCISES

6. Simplify each expression. These expressions are written in a conventional way.
   a. $2 \cdot 4^3$
   b. $-3 \cdot 2^4$
   c. $1 - 2^3 \cdot (-4)$

   Check that your calculator gives the same results.

parentheses can be used to get any desired order of operation

The best practice is to write things down so that it is clear which operations should be done first. You can specify the order of operations by using parentheses. Things inside parentheses are always done first. Thus, we have the memory device:

Please **E**xcuse **M**y **D**ear **A**unt **S**ally (PEMDAS)

Going from left to right:

Parentheses first (using PEMDAS, if needed, inside the parentheses)

Then **E**xponents.

Then **M**ultiplication/**D**ivision.

Finally **A**ddition/**S**ubtraction.

$(-2)^4$ *versus* -2^4

Now, we're ready to discuss the difference between $(-2)^4$ and -2^4.

Firstly,

$$(-2)^4 = (-2)(-2)(-2)(-2) = 16\,.$$

Next, rewrite -2^4 as $(-1) \cdot 2^4$, so that you can see that multiplication is competing with a power. Since the power wins, we have:

$$-2^4 = (-1) \cdot (2^4) = (-1) \cdot 16 = -16$$

Thus,

$$(-2)^4 = 16$$
$$-2^4 = -16$$

Different answers! Be careful about this! It is a common Algebra I mistake.

EXERCISES	7. Simplify each expression: a. $(-3)^2$ b. -3^2 c. $1-(-3)^2$ d. $1-3^2$ e. $1+(-3)^2$ f. $1+3^2$

negative integers n ;
a flip and repeated multiplication

Now let's talk about the situation with negative exponents. A negative exponent can get 'traded in for' a flip:

$$(\text{anything})^{-1} = \frac{1}{(\text{anything})^1} = \frac{1}{\text{anything}}$$

$$(\text{anything})^{-2} = \frac{1}{(\text{anything})^2}$$

$$(\text{anything})^{-3} = \frac{1}{(\text{anything})^3}\ ,\ \text{and so on.}$$

$(\frac{a}{b})^{-1} = \frac{b}{a}$

With fractions, it looks like this:

$$(\frac{a}{b})^{-1} = \frac{1}{(\frac{a}{b})^1} = \frac{1}{\frac{a}{b}} = 1 \div \frac{a}{b} = 1 \cdot \frac{b}{a} = \frac{b}{a}$$

Now that you've mired through this calculation once, you'll never have to do it this long way again. When a fraction is raised to the -1 power, the numerator becomes the new denominator, and the denominator becomes the new numerator. Here are examples:

$$(\frac{a}{b})^{-1} = \frac{b}{a}$$

$$(\frac{c}{d})^{-1} = \frac{d}{c}$$

$$(\frac{2}{5})^{-1} = \frac{5}{2}$$

$$(\frac{x+1}{y-2})^{-1} = \frac{y-2}{x+1}\ ,\ \text{and so on.}$$

EXERCISES	8. Simplify each expression:
	a. $(\frac{x}{y})^{-1}$
	b. $(\frac{1}{2})^{-1}$
	c. $(\frac{2}{3})^{-1}$
	d. $(\frac{x-1}{2y+3})^{-1}$

exponent laws

The 'exponent laws' give the tools for working with expressions involving exponents, and are discussed in the next section.

EXERCISES
web practice

Go to my homepage **https://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTIONS TO EXERCISES: EXPONENT NOTATION

1. a. the set of real numbers

b. 'Let ex be an element of arr' or 'Let ex be an element of the real numbers' or 'Let ex be a real number' or, most simply, 'Let ex be in arr'.

c. y is called the base; m is called the exponent or power.

d. 3 is called the base; 5 is called the exponent or power.

e. The integers are the set: $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

f. The positive integers are $1, 2, 3, \dots$.
The negative integers are $-1, -2, -3, \dots$.

g. True when $n = 7$ and $n = \frac{200}{5} = 40$; false when $n = \frac{1}{2}$ and $n = -10$.

h. $x^5 = x \cdot x \cdot x \cdot x \cdot x$

i. $y^0 = 1$

j. $y^{-3} = \frac{1}{y^3} = \frac{1}{y \cdot y \cdot y}$

k. $y^{-1} = \frac{1}{y^1} = \frac{1}{y}$

2. a. $(\text{positive})^{\text{even}}$ is positive

b. $(\text{positive})^{\text{odd}}$ is positive

c. $(\text{negative})^{\text{even}}$ is positive

d. $(\text{negative})^{\text{odd}}$ is negative

3. a. odd number of negative factors; result is negative

b. any number of positive factors is positive; result is positive

c. if x is negative, then $-x$ is positive; result is positive

d. if x is positive, then $-x$ is negative; odd number of negative factors; result is negative

e. if x is negative, then $-x$ is positive; result is positive

g. if x is positive, then $-x$ is negative; even number of negative factors is positive; result is positive

4. a. $(-3)^2 = 9$

b. $(-3)^3 = -27$

c. $1^{2008} = 1$

d. $(-1)^{2008} = 1$

e. $1^{2009} = 1$

f. $(-1)^{2009} = -1$

5. a. $1 + 3 \times 5 - 20 \div 4 = 1 + 15 - 5 = 11$

b. $-2 + 10 \div 5 \times 3 = -2 + 2 \times 3 = -2 + 6 = 4$

c. $4 - 3 + 12 \div 6 \times 2 + 1 = 4 - 3 + 2 \times 2 + 1 = 4 - 3 + 4 + 1 = 6$

6. a. $2 \cdot 4^3 = 2 \cdot 64 = 128$

b. $-3 \cdot 2^4 = -3 \cdot 16 = -48$

c. $1 - 2^3 \cdot (-4) = 1 - 8 \cdot (-4) = 1 - (8 \cdot (-4)) = 1 - (-32) = 1 + 32 = 33$

7. a. $(-3)^2 = 9$

b. $-3^2 = -9$

c. $1 - (-3)^2 = 1 - 9 = -8$

d. $1 - 3^2 = 1 - 9 = -8$

e. $1 + (-3)^2 = 1 + 9 = 10$

f. $1 + 3^2 = 1 + 9 = 10$

8. a. $(\frac{x}{y})^{-1} = \frac{y}{x}$

b. $(\frac{1}{2})^{-1} = \frac{2}{1} = 2$

c. $(\frac{2}{3})^{-1} = \frac{3}{2}$

d. $(\frac{x-1}{2y+3})^{-1} = \frac{2y+3}{x-1}$