19. MORE UNIT CONVERSION

Unit conversion problems are widespread. You'll encounter them not only in math classes, but also in chemistry, physics, on the SAT (Scholastic Aptitude Test), and when you need to exchange money in foreign countries. This section deals with more complicated unit conversion problems.

multi-step conversions

In the previous section, only 'one-step' conversions were presented: these require multiplying by 1 only once. Next, some 'multi-step' conversions are illustrated. These require multiplying by the number 1 more than once.

Recall this shorthand for repeated multiplication:

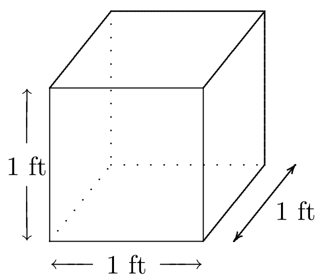
$$x^3 \quad \text{is a shorthand for} \quad x \cdot x \cdot x$$

The expression 'x^3' is read as 'x cubed' or 'x to the third power'.

Using this shorthand,

$$\text{'}\,\text{ft}^3\,\text{'} \quad \text{is another name for} \quad \text{'}\,(\text{ft})(\text{ft})(\text{ft})\,\text{'}$$

filling a sandbox



$(1\text{ ft})(1\text{ ft})(1\text{ ft})$
$= 1\text{ ft}^3$
$= 1$ cubic foot

Suppose that you need to buy sand to fill a sandbox. The sandbox is 4 feet wide, 7 feet long, and 1.5 feet deep. To completely fill it will require $(4)(7)(1.5) = 42$ cubic feet of sand. (One cubic foot is $(1\text{ ft})(1\text{ ft})(1\text{ ft}) = 1\text{ ft}^3$.) However, sand is sold in quantities of cubic yards (yd^3). How much sand should you buy?

Answer: We need to make the units of ft^3 (cubic feet) disappear, and make units of yd^3 (cubic yards) appear. Here we use the fact that $1\text{ yd} = 3\text{ ft}$, so that $\frac{1\text{ yd}}{3\text{ ft}}$ is a name for the number 1:

$$42\text{ ft}^3 = \overbrace{42\,\cancel{\text{ft}}^3}^{\text{original quantity}} \cdot \overbrace{\frac{1\text{ yd}}{3\,\cancel{\text{ft}}}}^{\text{multiply by 1 ...}} \cdot \overbrace{\frac{1\text{ yd}}{3\,\cancel{\text{ft}}}}^{\text{again ...}} \cdot \overbrace{\frac{1\text{ yd}}{3\,\cancel{\text{ft}}}}^{\text{and again!}} = \overbrace{\frac{42\cdot 1\cdot 1\cdot 1}{3\cdot 3\cdot 3}\text{ yd}^3}^{\text{exact answer}} \cong 1.6\text{ yd}^3\,.$$

Recall that the verb $\cong$ means 'is approximately equal to'. It isn't necessary to show multiplication by the number 1, so this could be more compactly written as:

$$42\text{ ft}^3 = 42\,\cancel{\text{ft}}^3\cdot\frac{1\text{ yd}}{3\,\cancel{\text{ft}}}\cdot\frac{1\text{ yd}}{3\,\cancel{\text{ft}}}\cdot\frac{1\text{ yd}}{3\,\cancel{\text{ft}}} = \frac{42}{3\cdot 3\cdot 3}\text{ yd}^3 \cong 1.6\text{ yd}^3\,.$$

a bigger sandbox

On the other hand, if we had an opportunity to buy 2.6 cubic yards of sand really cheap, and need to know if it's enough to fill a sandbox requiring 65 cubic feet, then we'd multiply by the number 1 in a different form:

$$2.6\text{ yd}^3 = \overbrace{2.6\,\cancel{\text{yd}}^3}^{\text{original quantity}} \cdot \overbrace{\frac{3\text{ ft}}{1\,\cancel{\text{yd}}}}^{\text{multiply by 1 ...}} \cdot \overbrace{\frac{3\text{ ft}}{1\,\cancel{\text{yd}}}}^{\text{again ...}} \cdot \overbrace{\frac{3\text{ ft}}{1\,\cancel{\text{yd}}}}^{\text{and again!}} = \frac{2.6\cdot 3\cdot 3\cdot 3}{1\cdot 1\cdot 1}\text{ ft}^3 = 70.2\text{ ft}^3\,.$$

It's enough! It isn't necessary to show division by the number 1, so this could be more compactly written as:

$$2.6\text{ yd}^3 = 2.6\,\cancel{\text{yd}}^3\cdot\frac{3\text{ ft}}{1\,\cancel{\text{yd}}}\cdot\frac{3\text{ ft}}{1\,\cancel{\text{yd}}}\cdot\frac{3\text{ ft}}{1\,\cancel{\text{yd}}} = 2.6\cdot 3\cdot 3\cdot 3\text{ ft}^3 = 70.2\text{ ft}^3\,.$$

EXERCISES

1. Suppose a sandbox is 5 ft by 5 ft by 1.5 ft . How many cubic yards of sand will be needed to completely fill it? Round your answer to 1 decimal place.
2. There's a sale, and you can buy 2 cubic yards of sand at a good price. Is this enough to fill a sandbox that holds 49 cubic feet?

units of volume:
yd^3
ft^3
in^3
etc.

Cubic yards (yd^3) and cubic feet (ft^3) are both units of volume, just as gallons, liters, and quarts are units of volume. Indeed, when you take any unit of length, and cube it, then you get a unit of volume. Thus, m^3 , in^3 , cm^3 , km^3 and mi^3 are all units of volume.

The volume units in the Unit Conversion Tables of the previous section are typically used for measures of liquid volume. However, volume is volume, whether it comes in solid or liquid form!

start to develop good calculator skills:

always get an exact answer first

When you are doing math problems that will eventually require some approximation (such as most unit conversion problems), you should always get an exact answer first—that is, resist the temptation to do *any* approximation until the last step. The next examples illustrate why.

the perils of approximating along the way

This is a very simple example that illustrates what can happen when you approximate along the way. A friend of mine recently visited Bulgaria. The current exchange rate was about 1.69 leva (a Bulgarian unit of currency) to one dollar. Let's just play a back-and-forth converting game, doing some approximation along the way:

Converting 2000 leva to dollars, and approximating to the nearest whole number gives:

$$2000 \text{ leva} \cdot \frac{\$1}{1.69 \text{ leva}} \cong \$1183$$

Then, converting these \$1183 back to leva, and approximating to the nearest whole number gives:

$$\$1183 \cdot \frac{1.69 \text{ leva}}{\$1} \cong 1999 \text{ leva}$$

One leva seems to have been lost in the process. Although this is a trivial example, it illustrates what can happen when approximations are made on intermediate steps.

Depending on the operations involved in your calculations, the sizes of the numbers being used, and even on the order that calculations are performed, approximations along the way can lead to a variety of different answers. Here's another example.

more perils:

three ways, three results

Suppose you're doing a calculation, and know that you want your final answer rounded to three decimal places. Here's the calculation, with three different names:

$$\frac{(57.2)(180.4)}{0.17} = \frac{57.2}{0.17} \cdot (180.4) = \frac{180.4}{0.17} \cdot (57.2)$$

Three different students do this calculation three different ways:

- $\frac{57.2}{0.17} \cong 336.471$; then $(336.471) \cdot (180.4) \cong 60699.368$
- $\frac{180.4}{0.17} \cong 1061.176$; then $(1061.176) \cdot (57.2) \cong 60699.267$
- Keying the entire calculation in without any intermediate rounding steps gives 60699.294 .

The three answers are close, but they're not the same. The only answer that is truly correct to 3 decimal places is the last one.

$169 \cdot \frac{27 \text{ cubic feet}}{1 \text{ cubic yard}}$

good calculator skills	Start developing good calculator skills now: • Always get an exact answer first. • Do the entire computation on your calculator without writing down any intermediate results. (You can store intermediate results, if needed.) • Approximate only at the last step. • When instructions say something like 'Round your answer to 3 decimal places' this means to round ONLY at the last step!

another advantage of getting an exact answer

There's another advantage of getting an exact answer. With an exact answer, you can produce an approximation to any number of decimal places, as needed.

However, if you only have a 3-decimal place approximation for a number, and then need 5 decimal places, you'd have to completely redo the computation.

format for simplifying expressions with an approximation

When you follow these guidelines for calculator usage, then the simplifying process will look like this, with an 'approximately equal to' sign only in the last step:

$$\begin{aligned} \textit{expression} &= \textit{rename} \\ &= \textit{rename} \\ &= \dots \\ &= \textit{still equal, finished renaming} \\ &\cong \textit{round to desired decimal place in the last step only} \end{aligned}$$

rates

A *rate* is a comparison of two quantities that are measured in different units.

Here are some examples of rates:

- $\frac{\$5}{\text{hr}}$, also commonly seen as \$5/hour, and read as '\$5 per hour'
- $\frac{40 \text{ mi}}{3 \text{ day}}$, read as '40 miles per 3 days'
- $\frac{10 \text{ kg}}{\text{in}^3}$, also commonly seen as 10 kg/in^3 , and read as '10 kilograms per cubic inch'

Multiplying by 1 to rename in an appropriate way provides a beautiful way to unify the solution of all kinds of rate problems.

EXERCISES	3. What is the definition of a *rate*? 4. Which of the following are rates? a. $\frac{3}{4}$ b. $\frac{1 \text{ cm}}{2 \text{ ft}}$ c. $\frac{3 \text{ cm}}{5 \text{ gal}}$

rates have lots of different names

A rate is a mathematical expression, and like all types of mathematical expressions, rates have lots of different names. For example,

$$\frac{\$5}{\text{hr}} = \frac{\$10}{2\text{ hr}} = \frac{\$2.50}{30\text{ min}} = \frac{50¢}{6\text{ min}}$$

Notice that each of these names has a unit of currency (a money unit) in the numerator, and a unit of time in the denominator.

all names for the same rate must have the same type of unit in both numerator and denominator

A rate can be renamed to a new desired name, providing the *type* of units in the numerator and denominator (e.g., length, time, volume, mass/weight) remain the same.

For example, suppose a rate has a unit of length in the numerator, and a unit of volume in the denominator. Then, it can only be renamed to rates that have length in the numerator and volume in the denominator.

EXERCISES

5. Suppose a rate is of the form $\frac{\text{ft}}{\text{gal}}$. Which of the following forms could it be renamed in?
   a. $\frac{\text{mi}}{\text{liter}}$
   b. $\frac{\text{ft}}{\text{in}^3}$
   c. $\frac{\text{sec}}{\text{gal}}$
   d. $\frac{\text{ft}}{\text{tb}}$
   e. $\frac{\text{cm}}{\text{tsp}^3}$
   f. $\frac{\text{km}}{\text{quart}}$

get used to these!

Before looking at a typical rate problem, you need to get over any discomfort with expressions like this:

$$\frac{\frac{1}{3}}{\frac{1}{3}} \quad \text{or} \quad \frac{\frac{1}{2.4}}{\frac{1}{2.4}} \quad \text{or} \quad \frac{\frac{1}{5280}}{\frac{1}{5280}}$$

compound fractions

These have 'fractions within fractions' and are sometimes called *compound fractions*. These particular compound fractions all happen to be names for the number 1 , since the numerator and denominator are equal.

Compound fractions are pretty ugly-looking, and are often avoided. However, when used properly in rate problems, they'll allow us to do just what we need to do in a very efficient way, as you'll see in the following paragraphs.

a typical rate problem

Here's a typical rate problem:

A snail crawls about 2 feet in one hour.

How many minutes will it take to crawl 7 inches?

The solution will go like this:

- Step 1: Identify the original rate.
- Step 2: Identify the name for the rate that you *want*. Use x for any number that is unknown.
- Step 3: Check that the original rate and its desired new name are 'compatible'—same type of units in both numerator and denominator.
- Step 4: Turn the original rate into the new name by multiplying by 1 in appropriate ways.

the solution

Here's the solution, step-by-step.

- Step 1: The original rate is $\frac{2 \text{ ft}}{1 \text{ hr}}$.
- Step 2: The new desired name is $\frac{7 \text{ in}}{x \text{ min}}$.
- Step 3: Both rates have units of length upstairs (feet and inches), and units of time downstairs (hours and minutes). They're compatible.
- Step 4: The renaming goes like this. Don't despair! This renaming is discussed in more detail in the next paragraph. After a little practice, you'll be solving problems like this with ease, confidence, and efficiency. The lines are labeled for easy reference in the discussion that follows.

$$\begin{aligned}
\frac{2 \text{ ft}}{1 \text{ hr}} &= \frac{2 \text{ ft}}{1 \text{ hr}} \cdot \frac{12 \text{ in}}{1 \text{ ft}} \cdot \frac{1 \text{ hr}}{60 \text{ min}} && \text{(a)} \\
&= \frac{24 \text{ in}}{60 \text{ min}} && \text{(b)} \\
&= \frac{24 \text{ in}}{60 \text{ min}} \cdot \frac{\frac{1}{24}}{\frac{1}{24}} \cdot \frac{7}{7} && \text{(c)} \\
&= \frac{7 \text{ in}}{60 \cdot \frac{1}{24} \cdot 7 \text{ min}} && \text{(d)} \\
&= \frac{7 \text{ in}}{17.5 \text{ min}} && \text{(e)}
\end{aligned}$$

It takes 17.5 minutes for the snail to crawl 7 inches.

the step-by-step discussion

(a) Always take care of the units first.

To change from feet to inches in the numerator, multiply by $\frac{12 \text{ in}}{1 \text{ ft}}$.

To change from hours to minutes in the denominator, multiply by $\frac{1 \text{ hr}}{60 \text{ min}}$.

Cross out all the units that cancel, and circle the units that survive (which must be the desired units).

(b) Group together all the numbers in the numerator, and write these in front of the unit.

Group together all the numbers in the denominator, and write these in front of the unit.

If the numbers are easy to multiply together (as they were in this example) then do the multiplication.

(c) Now, look at the number you WANT in your new name. You must always know where you're heading, or you'll never get there!

We want a 7 in the numerator.

Right now we have a 24 in the numerator. We need to get rid of the 24, and bring in the 7.

To get rid of the 24, multiply by $\frac{\frac{1}{24}}{\frac{1}{24}}$. Cancel it out in the numerator. This is the part that seems most troublesome to people using this technique for the first time. Rest assured though that after doing it a few times, it will become easy and natural.

To bring in the 7, multiply by $\frac{7}{7}$. Circle the 7 in the numerator.

(d) Rewrite the numerator, which is now precisely the numerator you want.

Gather together the numbers in the denominator. These aren't easy to multiply together in your head, so leave them to do on your calculator.

(e) Use your calculator to multiply together the numbers in the denominator. If the answer you report is exact, put an 'equal' sign. If the answer you report is approximate, put an 'approximately equal to' sign.

EXERCISES

6. Redo the snail problem, without looking at your notes or your text. If you get stuck, then look back to see what to do. But then close your book and try it again yourself. Repeat this problem until you can do the entire snail problem correctly, without looking at anything.

7. Redo the snail problem again, but this time write your original rate as $\frac{1 \text{ hr}}{2 \text{ ft}}$.

8. Now try this one:

   A marathon crawler snail covers 6 feet in one hour.

   How long will it take to cover 26.2 feet?

more compact when you have more experience

The author has been doing rate problems like this for a long time. Consequently, she writes things down much more compactly, like this:

$$\frac{2 \text{ ft}}{1 \text{ hr}} = \frac{2 \text{ ft}}{1 \text{ hr}} \cdot \frac{12 \text{ in}}{1 \text{ ft}} \cdot \frac{1 \text{ hr}}{60 \text{ min}} \cdot \frac{\frac{1}{24}}{\frac{1}{24}} \cdot \frac{7}{7} = \frac{7 \text{ in}}{17.5 \text{ min}}$$

As you gain experience with problems like these, feel free to write things down more compactly.

an important thing to remember in rate problems

When solving rate problems, be sure to take care of all your unit conversions FIRST. Then, take care of any particular number you want LAST. The unit conversions will likely bring additional numbers into the picture that you will have to deal with.

EXERCISES

9. The web has unlimited practice with rate problems. They all look something like the ones given here.

   If you can give an exact answer in 3 or fewer decimal places, then do so. Otherwise, round to 3 decimal places.

   Suppose that an object travels 5 feet in 8 minutes.

   a. How many feet will it travel in 20 minutes?
   b. How many feet will it travel in 2 hours?
   c. How many hours will it take to travel 107 feet?
   d. How many minutes will it take to travel 43 centimeters?

'passing cars' problem

To complete this section, we return to the 'passing cars' problem mentioned earlier in the text:

> The author of this book usually drives the speed limit. Consequently, she often finds herself being passed by other cars on the freeway. Just how fast *are* those other cars going when they whiz by?

My car is about 14 feet long. Suppose it takes about 2 seconds for the front of a passing car to travel from my rear bumper to my front bumper. Then, the passing car's additional speed is $\frac{14 \text{ ft}}{2 \text{ sec}}$. How many miles per hour is that?

$$\frac{14 \text{ ft}}{2 \text{ sec}} = \frac{14 \not{\text{ft}}}{2 \not{\text{sec}}} \cdot \frac{1 \text{ mile}}{5280 \not{\text{ft}}} \cdot \frac{60 \not{\text{sec}}}{1 \not{\text{min}}} \cdot \frac{60 \not{\text{min}}}{1 \text{ hr}} = \frac{14 \cdot 60 \cdot 60}{2 \cdot 5280} \frac{\text{miles}}{\text{hr}} \cong 4.8 \text{ mph}$$

The passing car is going about five miles/hour faster than the author.

EXERCISE

10. Then there's the car that whizzes by in about 0.5 seconds (that is, half a second). How much faster than my car is it going?

EXERCISES
web practice

Go to my homepage **https://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTIONS TO EXERCISES: MORE UNIT CONVERSION

1. $5\cdot 5\cdot 1.5 = 37.5$; $37.5\ \text{ft}^3 = 37.5\ \text{ft}^3 \cdot \frac{1\ \text{yd}}{3\ \text{ft}} \cdot \frac{1\ \text{yd}}{3\ \text{ft}} \cdot \frac{1\ \text{yd}}{3\ \text{ft}} \cong 1.4\ \text{yd}^3$; you'll need about 1.4 cubic yards of sand.

2. $2\ \text{yd}^3 = 2\ \text{yd}^3 \cdot \frac{3\ \text{ft}}{\text{yd}} \cdot \frac{3\ \text{ft}}{\text{yd}} \cdot \frac{3\ \text{ft}}{\text{yd}} = 54\ \text{ft}^3$; it's enough!

3. A rate is a comparison of two quantities that are measured in different units.

4. (b) and (c) are rates

5. (a), (b), (d), (f)

7.

$$
\begin{aligned}
\frac{1\ \text{hr}}{2\ \text{ft}} &= \frac{1\ \text{hr}}{2\ \text{ft}} \cdot \frac{1\ \text{ft}}{12\ \text{in}} \cdot \frac{60\ \text{min}}{1\ \text{hr}} \\
&= \frac{60\ \text{min}}{24\ \text{ft}} \\
&= \frac{60\ \text{min}}{24\ \text{ft}} \cdot \frac{\frac{1}{24}}{\frac{1}{24}} \cdot \frac{7}{7} \\
&= \frac{60 \cdot \frac{1}{24} \cdot 7\ \text{min}}{7\ \text{in}} \\
&= \frac{17.5\ \text{min}}{7\ \text{in}}
\end{aligned}
$$

8.

$$
\begin{aligned}
\frac{6\ \text{ft}}{1\ \text{hr}} &= \frac{6\ \text{ft}}{1\ \text{hr}} \cdot \frac{1\ \text{hr}}{60\ \text{min}} \\
&= \frac{6\ \text{ft}}{60\ \text{min}} = \frac{1\ \text{ft}}{10\ \text{min}} \\
&= \frac{1\ \text{ft}}{10\ \text{min}} \cdot \frac{26.2}{26.2} \\
&= \frac{26.2\ \text{ft}}{10 \cdot 26.2\ \text{min}} \\
&= \frac{26.2\ \text{ft}}{262\ \text{min}}
\end{aligned}
$$

It will take the snail 262 minutes to cover 26.2 feet.

9. These solutions are written very compactly.

a.

$$\frac{5\text{ ft}}{8\text{ min}} = \frac{5\text{ ft}}{8\text{ min}} \cdot \frac{\frac{1}{8}}{\frac{1}{8}} \cdot \frac{20}{20} = \frac{12.5\text{ ft}}{20\text{ min}}$$

The object can travel 12.5 feet in 20 minutes.

b.

$$\frac{5\text{ ft}}{8\text{ min}} = \frac{5\text{ ft}}{8\text{ min}} \cdot \frac{60\text{ min}}{1\text{ hr}} \cdot \frac{\frac{1}{8}}{\frac{1}{8}} \cdot \frac{2}{2} = \frac{75\text{ ft}}{2\text{ hr}}$$

The object can travel 75 feet in 2 hours.

c.

$$\frac{5\text{ ft}}{8\text{ min}} = \frac{5\text{ ft}}{8\text{ min}} \cdot \frac{60\text{ min}}{1\text{ hr}} \cdot \frac{\frac{1}{300}}{\frac{1}{300}} \cdot \frac{107}{107} \cong \frac{107\text{ ft}}{2.853\text{ hr}}$$

It will take about 2.853 hours to cover 107 feet.

d.

$$\frac{5\text{ ft}}{8\text{ min}} = \frac{5\text{ ft}}{8\text{ min}} \cdot \frac{12\text{ in}}{1\text{ ft}} \cdot \frac{2.5\text{ cm}}{1\text{ in}} \cdot \frac{\frac{1}{60}}{\frac{1}{60}} \cdot \frac{\frac{1}{2.5}}{\frac{1}{2.5}} \cdot \frac{43}{43} \cong \frac{43\text{ cm}}{2.293\text{ min}}$$

It will take about 2.293 minutes to cover 43 centimeters.

10.

$$\frac{14\text{ ft}}{0.5\text{ sec}} = \frac{14\text{ ft}}{0.5\text{ sec}} \cdot \frac{1\text{ mi}}{5280\text{ ft}} \cdot \frac{60\text{ sec}}{1\text{ min}} \cdot \frac{60\text{ min}}{1\text{ hr}} = \frac{14 \cdot 60 \cdot 60\text{ mi}}{0.5 \cdot 5280\text{ hr}} \cong 19\text{ mph}$$

The car is going about 19 miles per hour faster than me.