13. ADDING FRACTIONS

a sample addition problem

Fractions have lots of different names, and renaming is often needed to add or subtract fractions. Here's how you'd add $\frac{1}{2}$ and $\frac{1}{3}$:

$$\begin{aligned} \frac{1}{2} + \frac{1}{3} &= \frac{1}{2} \cdot \frac{3}{3} + \frac{1}{3} \cdot \frac{2}{2} \\ &= \frac{3}{6} + \frac{2}{6} \quad \text{(they now have a common denominator)} \\ &= \frac{5}{6} \end{aligned}$$

The next few paragraphs discuss the various ideas in this problem.

adding fractions with a common denominator

When fractions have the same denominator (called a *common denominator*), it's easy to add them. For all real numbers A and B , and for $C \neq 0$,

$$\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C} \,. \tag{1}$$

You may recognize this property from the **Introduction to Fractions** section, where it was noted that

$$\frac{A+B}{C} = \frac{A}{C} + \frac{B}{C} \,. \tag{2}$$

Every equation works in two directions! Version (1) of this equation gives the rule for adding fractions that have a common denominator. Version (2) of this equation gives the rule for rewriting a fraction with a sum in the numerator.

EXERCISES

1. Add the following fractions:
   a. $\frac{1}{3} + \frac{4}{3}$
   b. $\frac{4}{x+3} + \frac{9}{x+3}$
   c. $\frac{1.3}{7.92} + \frac{2.4}{7.92}$
   d. $\frac{1}{7} + \frac{2}{7} + \frac{4}{7}$

common denominators

When fractions to be added don't initially have a common denominator, you need to find one, and then rewrite the fractions with this common denominator.

For a common denominator, you need a number that each of the individual denominators goes into evenly.

For example, to add $\frac{1}{2}$ and $\frac{1}{3}$ you'd probably use a denominator of 6 , because 2 goes into 6 evenly, and 3 goes into 6 evenly, and 6 is the smallest number that both go into evenly.

As a second example, to add $\frac{1}{6}$ and $\frac{5}{12}$, you'd probably use a denominator of 12 , because 6 and 12 both go into 12 evenly, and 12 is the smallest number with this property.

The next few paragraphs discuss the process of finding a good common denominator.

multiples

The *multiples* of 2 are 2, 4, 6, 8, 10, 12, 14, 16, 18, ...

Notice that the multiples of 2 are obtained by taking the number 2, and multiplying successively by 1, 2, 3, ...

The *multiples* of 3 are 3, 6, 9, 12, 15, 18, ...

Notice that the multiples of 3 are obtained by taking the number 3, and multiplying successively by 1, 2, 3, ...

In general, the multiples of a number x are:

$$x,\ 2x,\ 3x,\ 4x,\ \ldots\ .$$

EXERCISES

2. a. What are the multiples of 4?
   b. What are the multiples of 10?

least common multiple

In the previous paragraph, you saw that the multiples of 2 and 3 are:

$$\begin{aligned}\text{multiples of } 2: &\quad 2, 4, 6, 8, 10, 12, 14, 16, 18, \ldots\\ \text{multiples of } 3: &\quad 3, 6, 9, 12, 15, 18, \ldots\end{aligned}$$

What numbers are multiples of *both* 2 and 3? That is, what numbers appear in *both* lists above? Answer: 6, 12, 18 and so on.

What is the *least* number that is a multiple of *both* 2 and 3? Answer: 6.

The number 6 is called the *least common multiple* of 2 and 3, because it is a common multiple (i.e., it is a multiple of 2 and a multiple of 3), and it is the smallest number with this property.

To find a common denominator, you usually want to use the least common multiple of the individual denominators.

the list method for finding a least common multiple

If the individual denominators aren't too big, the quickest way to come up with the least common multiple is to mentally go through the multiples of the largest number, checking each to see if it is a multiple of *all* the numbers.

For example, suppose that you need to add these three fractions: $\frac{1}{3}$, $\frac{2}{5}$, and $\frac{7}{20}$. The individual denominators are 3, 5, and 20. For a common denominator, you want the least common multiple of 3, 5, and 20.

Go through the multiples of 20, stopping to check if each is divisible by the other two numbers, 3 and 5.

Is 20 divisible by both 3 and 5? No; it's not divisible by 3.

Is 40 divisible by both 3 and 5? No; it's not divisible by 3.

Is 60 divisible by both 3 and 5? Yes! So, 60 is the least common multiple.

Finishing the problem:

$$\begin{aligned}\frac{1}{3}+\frac{2}{5}+\frac{7}{20} &= \frac{1}{3}\cdot\frac{20}{20}+\frac{2}{5}\cdot\frac{12}{12}+\frac{7}{20}\cdot\frac{3}{3}\\ &= \frac{20}{60}+\frac{24}{60}+\frac{21}{60}\\ &= \frac{20+24+21}{60}\\ &= \frac{65}{60}\,.\end{aligned}$$

Don't worry for the moment about simplifying your answers.

EXERCISES

3. Use the list method to find the least common multiple of each set of numbers.
   a. 3 and 5
   b. 6 and 10
   c. 2, 5, and 12
   d. 20 and 25

fractions have lots of different names

Notice that once you know the desired common denominator, the renaming is accomplished by multiplying by 1 in an appropriate form. Multiplying by 1 doesn't change a number (i.e., doesn't change its location on the real number line), but it does give a new name for the number.

For example, suppose that the fraction $\frac{2}{5}$ must be renamed with a denominator of 60. Here's the thought process:

What number, times 5, gives 60? That is, how many times does 5 go into 60? Answer: 12.

$$\frac{2}{5} \cdot \frac{x}{x} = \frac{?}{60}$$

Thus,

$$\frac{2}{5} \cdot \frac{12}{12} = \frac{24}{60}\ .$$

shortcut for renaming fractions

Some people like to use the following shortcut for renaming fractions:

$$\frac{2 \rightarrow 24}{5 \rightarrow 60}$$

- Step 1: 5 goes into 60 how many times? 12
- Step 2: 12 times 2 is 24.
- Step 3: The new numerator is 24.

EXERCISES

4. Rename each fraction with the specified denominator:
   a. $\frac{2}{3}$ with a denominator of 30
   b. $\frac{1}{8}$ with a denominator of 24
   c. $\frac{7}{4}$ with a denominator of 12
   d. $\frac{1}{10}$ with a denominator of 50

another method for finding the least common multiple

The list method for finding the least common multiple isn't always practical. If you find yourself checking too many numbers, then there's a better way. The concepts discussed next will also be important when fractions with variables are explored later in the course.

factors: numbers that go in evenly

In the next few paragraphs, 'number' refers to a counting number: $1, 2, 3, \ldots$

The *factors* of a number are the numbers that go into it evenly.

For example, the factors of 10 are 1, 2, 5, and 10.

The factors of 42 are 1, 2, 3, 6, 7, 14, 21, and 42.

factors occur in pairs

Notice that factors occur in pairs:

$1 \cdot 42 = 42$

$2 \cdot 21 = 42$

$3 \cdot 14 = 42$

$6 \cdot 7 = 42$

every number has 1 and itself as factors

Notice that every number has a factor of 1, because 1 goes into everything evenly. Also, every number has itself as a factor.

prime numbers: 2, 3, 5, 7, 11, 13, . . .

A number greater than 1 is called *prime* if the only numbers that go into it evenly are itself and 1. The first few prime numbers are 2, 3, 5, 7, 11, and 13.

★ *efficiency in determining if a number is prime*

The factors of a number N always occur in pairs, where one factor in the pair is less than or equal to $\sqrt{N}$, and the other factor is greater than or equal to $\sqrt{N}$.

For example, $4 \cdot 11 = 44$: $4 < \sqrt{44}$ and $11 > \sqrt{44}$.

To determine if a number is prime, it is only necessary to check the prime factors that are less than or equal to its square root: if a number has a factor other than itself or 1, then it has a prime factor.

For example, to determine if 701 is prime, it is only necessary to check the prime factors less than $\sqrt{701} \approx 26.5$.

The number 701 is not divisible by 2, 3, 5, 7, 11, 13, 17, 19, or 23, so it is prime.

EXERCISES

5. List the factors of each number. If the number is prime, so state.
   a. 12
   b. 17
   c. 9
   d. 28

factoring a number into its primes

Every counting number can be written as a unique product of prime factors; this is called the *prime factorization*.

To find the prime factorization, just start by writing the number as *any* product that jumps into your head, and keep breaking it down until you have all primes. When you're done, report the result is a nice organized fashion, using exponents and writing the prime factors in increasing order. For example, here's how you might get the prime factorization of the number 200:

$$\begin{aligned} 200 &= 20 \cdot 10 \\ &= 4 \cdot 5 \cdot 2 \cdot 5 \\ &= 2 \cdot 2 \cdot 5 \cdot 2 \cdot 5 \\ &= 2^3 \cdot 5^2 \end{aligned}$$

Someone else might go like this:

$$\begin{aligned} 200 &= 2 \cdot 100 \\ &= 2 \cdot 10 \cdot 10 \\ &= 2 \cdot 2 \cdot 5 \cdot 2 \cdot 5 \\ &= 2^3 \cdot 5^2\,. \end{aligned}$$

No matter how you do it, you'll always get to the same place.

EXERCISES	6. Find the prime factorization for each of the following numbers: a. 150 b. 500 c. 340 d. 144

finding the least common multiple from the prime factorizations:

least common M*ultiple;* M*ost*

The least common Multiple is easy to find from the prime factorizations: just take the Most number of times that each factor appears. For example, suppose you need to find the least common multiple of 840 and 4,950 . The list method is *not* practical here! But it's not bad with the prime factorization method. First, find the prime factorizations:

$$\begin{aligned} 840 &= 2^3 \cdot 3 \ \ \cdot 5 \ \ \cdot 7 \\ 4{,}950 &= 2 \ \ \cdot 3^2 \cdot 5^2 \cdot \qquad 11 \end{aligned}$$

The prime factorizations involve factors of 2, 3, 5, 7, and 11 .

What's the *most* number of times that 2 appears? It appears 3 times in the factorization for 840 . It appears 1 time in the factorization of 4,950 . The most number of times it appears is 3 .

What's the *most* number of times that 3 appears? That is, what is the highest exponent on 3 ? Answer: 2

What's the *most* number of times that 5 appears? That is, what is the highest exponent on 5 ? Answer: 2

What's the *most* number of times that 7 appears? Answer: 1

What's the *most* number of times that 11 appears? Answer: 1

Thus, the least common multiple is $2^3 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 = 138{,}600$.

EXERCISES	7. Use the prime factorization method to find the least common multiple of: a. 4, 10, and 35 b. 260 and 1,020

subtracting fractions

Since subtraction is just addition of the opposite, there are no new ideas to subtract fractions. Here's an example that uses both addition and subtraction:

$$\begin{aligned} \frac{1}{5} - \frac{2}{7} + \frac{8}{15} &= \frac{1}{5}\cdot\frac{21}{21} - \frac{2}{7}\cdot\frac{15}{15} + \frac{8}{15}\cdot\frac{7}{7} \\ &= \frac{21}{105} - \frac{30}{105} + \frac{56}{105} \\ &= \frac{21-30+56}{105} \\ &= \frac{47}{105} \end{aligned}$$

EXERCISES	8. Why do you suppose the author named this section 'Adding Fractions' instead of 'Adding and Subtracting Fractions'?

unifying several ideas

An astute reader may have noticed a connection between several ideas discussed thus far in this text. Read through the following sentences:

The number 38 is divisible by 2 .

The number 38 is a multiple of 2 .

The number 2 is a factor of 38 .

The number 2 goes into 38 evenly.

Each one of these sentences gives exactly the same information, just in a different way. If, for example, you know that 38 is divisible by 2 , then you also know that 38 is a multiple of 2 ; that 2 is a factor of 38 ; and that the number 2 goes into 38 evenly.

equivalent sentences

More generally, consider the following sentences:

The number N is divisible by k .

The number N is a multiple of k .

The number k is a factor of N .

The number k goes into N evenly.

Given values of N and k , if one of these sentences is true, then they are all true.

Given values of N and k , if one of these sentences is false, then they are all false.

These four sentences are completely interchangeable—they are true and false at the same time. In the language of math, we say that these four sentences are 'equivalent'. The idea of equivalence is discussed in detail in a future section.

EXERCISES

9. Decide if each of the following sentences is true or false. Then, state the sentence in three different ways.
   a. 56 is a multiple of 7
   b. 3 goes into 17 evenly

EXERCISES
web practice

Go to my homepage **https://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTIONS TO EXERCISES: ADDING FRACTIONS

1. a. $\frac{1}{3}+\frac{4}{3}=\frac{5}{3}$

b. $\frac{4}{x+3}+\frac{9}{x+3}=\frac{13}{x+3}$

c. $\frac{1.3}{7.92}+\frac{2.4}{7.92}=\frac{3.7}{7.92}$

d. $\frac{1}{7}+\frac{2}{7}+\frac{4}{7}=\frac{7}{7}=1$

2. a. The multiples of 4 are: $4, 8, 12, 16, \ldots$.

b. The multiples of 10 are: $10, 20, 30, 40, \ldots$.

3. a. Go through the multiples of 5, checking if they're divisible by 3: 5, 10, 15; 15 is the least common of 3 and 5.

b. Go through the multiples of 10, checking if they're divisible by 6: 10, 20, 30; 30 is the least common multiple of 6 and 10.

c. Go through the multiples of 12, checking if they're divisible by both 2 and 5: 12, 24, 36, 48, 60; 60 is the least common multiple of 2, 5, and 12.

d. Go through the multiples of 25, checking if they're divisible by 20: 20, 40, 60, 80, 100; 100 is the least common multiple of 20 and 25.

4. a. $\frac{2}{3}=\frac{20}{30}$

b. $\frac{1}{8}=\frac{3}{24}$

c. $\frac{7}{4}=\frac{21}{12}$

d. $\frac{1}{10}=\frac{5}{50}$

5. a. the factors of 12 are 1, 2, 3, 4, 6, and 12

b. the only factors of 17 are 1 and 17; 17 is a prime number

c. the factors of 9 are 1, 3, and 9

d. the factors of 28 are 1, 2, 4, 7, 14, and 28

6. a. $150=2\cdot 3\cdot 5^2$

b. $500=2^2\cdot 5^3$

c. $340=2^2\cdot 5\cdot 17$

d. $144=2^4\cdot 3^2$

7. a. 4, 10, and 35

$4=2^2$

$10=2\cdot 5$

$35=5\cdot 7$

The least common multiple is $2^2\cdot 5\cdot 7=140$.

b. $260=2^2\cdot 5\cdot 13$

$1020=2^2\cdot 3\cdot 5\cdot 17$

The least common multiple is $2^2\cdot 3\cdot 5\cdot 13\cdot 17=13{,}260$.

8. To a mathematician, there is only addition, since every subtraction problem is an addition problem in disguise: $a-b=a+(-b)$.

9. a. The sentence '56 is a multiple of 7' is true.

Three different ways to express this same (true) idea:

56 is divisible by 7

7 is a factor of 56

7 goes into 56 evenly

b. The sentence '3 goes into 17 evenly' is false.

Three different ways to express this same (false) idea:

3 is a factor of 17

17 is a multiple of 3

17 is divisible by 3

