GEOMETRY OBJECTIVE: LOGIC4

equivalent forms of conditional sentences (implications):

- If A, then B (or: B, if A)
- A implies B
- $A \Rightarrow B$
- Whenever A, B (or: B, whenever A)

hypothesis; conclusion; vacuously true; converse; contrapositive

DISCUSSION OF CONCEPTS:

A sentence of the form "$A \Rightarrow B$" is called an *implication* or a *conditional sentence*. Recall from LOGIC3 that A is called the *hypothesis*, B is called the *conclusion*, and the truth table for an implication is:

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

When the hypothesis of an implication is false (lines 3 and 4 of the truth table), then the implication is true; it is said to be *vacuously true* in this situation. For example, the sentence

"If $1 = 2$, then $3 = 4$."

is vacuously true. (Perhaps think of it this way: on the day when 1 equals 2, it will also be true that 3 equals 4. It just isn't ever going to happen!)

HOW TO PROVE THAT AN IMPLICATION IS TRUE (DIRECT PROOF):

Here's why we define an implication to be true when its hypothesis is false:

To prove that an implication is true, all you need to show is that whenever the hypothesis is true, the conclusion is also true (line 1). You don't ever need to worry about what happens when the hypothesis is false, because the implication is vacuously true in this situation!

When you prove that an implication is true by assuming that the hypothesis is true, and then showing that the conclusion is also true, this is called a **DIRECT PROOF**.

So, suppose you're asked to prove that the sentence

"If $x > 3$, then $x > 2$."

is true. Here's what you'd do:

- Suppose that the hypothesis is TRUE; i.e., suppose that x is greater than 3. Thus, x is a a number that lies to the right of 3 on a number line.
- Show that the conclusion is TRUE: If x lies to the right of 3, then it must also lie to the right of 2. Thus, the sentence $x > 2$ is also true.
- In summary: whenever a number is greater than 3, then it is also greater than 2. Whenever the hypothesis is true, the conclusion is also true. Thus, the implication is true.

(BY THE WAY: When a mathematician talks about the sentence "If $x > 3$, then $x > 2$." they are REALLY talking about the sentence: "For all x, if $x > 3$, then $x > 2$." Thus, for the sentence to be true, it must be true for ALL values of x. Students probably don't need to worry too much about this, at this point... but teachers should have this higher level of understanding.)

PROVING THAT AN IMPLICATION IS FALSE: A COUNTEREXAMPLE:

To prove that an implication is false, you must produce a *specific example* for which the hypothesis is true, but the conclusion is false. Such an example is called a **counterexample**.

For example, to prove that the sentence

$$\text{“If } x > 2\text{, then } x > 3\text{.”}$$

is false, you'd have to produce a number that is greater than 2 (the hypothesis is true), but that is not greater than 3 (the conclusion is false). There are many numbers that will work: choose, for example, 2.5. You could report your answer like this:

COUNTEREXAMPLE: Let $x = 2.5$. Then, the hypothesis ($2.5 > 2$) is true, but the conclusion ($2.5 > 3$) is false.

Here's a slightly more elegant way to state the counterexample:

Let $x = 2.5$. Then, $2.5 > 2$, but $2.5 \not> 3$.

Here's an even *more* elegant way to state the counterexample, which requires some knowledge about negating sentences (LOGIC6):

Let $x = 2.5$. Then, $2.5 > 2$, but $2.5 \le 3$.

EQUIVALENT FORMS OF IMPLICATIONS:

The importance of a mathematical sentence type is perhaps best measured by how many different ways you can say the same thing. (Mathematicians, like other people, tend to get bored always saying things the same way.) There are many equivalent forms of an implication, the most common of which are listed below. Even though all these sentences *look* different, they are equivalent: that is, they are true at the same time, and false at the same time; they are completely interchangeable:

EQUIVALENT FORMS OF IMPLICATIONS

$A \Rightarrow B$

If A, then B

B, if A

A implies B

Whenever A, (then) B

B, whenever A

For example, all the following are equivalent:

$(x > 3) \Rightarrow (x > 2)$

If $x > 3$, then $x > 2$

$x > 2$, if $x > 3$

$x > 3$ implies $x > 2$

Whenever $x > 3$, then $x > 2$

$x > 2$, whenever $x > 3$

CONVERSE and CONTRAPOSITIVE:

Given an implication, there are two new sentences that can be formed: its *converse*, and its *contrapositive*. Here are the definitions of these new sentences.

CONVERSE OF AN IMPLICATION:

Let “$A \Rightarrow B$” be an implication. Switching the roles of the hypothesis and conclusion gives a new sentence: “$B \Rightarrow A$”. The sentence “$B \Rightarrow A$” is called the **CONVERSE** of the sentence “$A \Rightarrow B$”.

The truth table below shows that the sentences “$A \Rightarrow B$” and “$B \Rightarrow A$” are NOT equivalent. They can have different truth values. An implication is NOT equivalent to its converse.

A	B	$A \Rightarrow B$	$B \Rightarrow A$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

For example, consider the sentence "If it's raining, then the ground is wet." This sentence is true. Its converse is: "If the ground is wet, then it's raining." This sentence is false; the ground could be wet because someone dumped a bucket of water on it.

CONTRAPOSITIVE OF AN IMPLICATION:

Let " $A \Rightarrow B$ " be an implication. The sentence " (not B) $\Rightarrow$ (not A) " is called the **CONTRAPOSITIVE** of the implication.

For example, the contrapositive of the sentence "If it's raining, then the ground is wet." is "If the ground isn't wet, then it isn't raining."

The truth table below shows that *an implication is equivalent to its contrapositive.* Thus, an implication and its contrapositive can be used completely interchangeably; they are true at the same time and false at the same time. Often, when one needs to prove an implication, they instead work with its contrapositive.

A	B	not B	not A	$A \Rightarrow B$	(not B) $\Rightarrow$ (not A)
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

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SAMPLE PROBLEMS and SOLUTIONS

Write each implication below in five equivalent ways. Decide if the implication is true or false. If it is false, give a counterexample.

Then, write both the converse and the contrapositive of the implication.

1. If $x > 4$, then $x \geq 5$.
2. If $x > 0$, then $x^2 > 0$.
3. If $x^2 > 0$, then $x > 0$.
4. If $x < 0$, then $|x| < 0$.
5. If $x < 0$, then $|x| > 0$.
6. If $|x| > 0$, then $x > 0$.
7. If $x \in (1, 4)$, then $x \in (2, 3)$.
8. If $x \in (1, 2)$, then $x \in (1, 2]$.
9. If $x \in (1, 2]$, then $x \in (1, 2)$.
10. If $1 < x < 2$, then $1 < |x| < 2$.

PARTIAL SOLUTIONS:

1. If $x > 4$, then $x \geq 5$.

$(x > 4) \Rightarrow (x \geq 5)$

$x \geq 5$, if $x > 4$

$x > 4$ implies $x \geq 5$

Whenever $x > 4$, then $x \geq 5$.

$x \geq 5$, whenever $x > 4$

The implication is TRUE.

CONVERSE: If $x \geq 5$, then $x > 4$.

CONTRAPOSITIVE: If $x \ngeq 5$, then $x \ngtr 4$.

(More elegant: If $x < 5$, then $x \leq 4$.)

2. The implication is TRUE.

3. The implication is FALSE.

COUNTEREXAMPLE: Let $x = -2$. Then, the hypothesis ($(-2)^2 > 0$) is TRUE, but the conclusion ($-2 > 0$) is FALSE.

4. The implication is FALSE.

COUNTEREXAMPLE: Let $x = -2$. Then, the hypothesis ($-2 < 0$) is TRUE, but the conclusion ($|-2| < 0$) is FALSE.

5. The implication is TRUE.

6. The implication is FALSE.

COUNTEREXAMPLE: Let $x = -2$. Then, the hypothesis ($|-2| > 0$) is TRUE, but the conclusion ($-2 > 0$) is FALSE.

7. The implication is TRUE.

8. The implication is TRUE.

9. The implication is FALSE.

COUNTEREXAMPLE: Let $x = 2$. Then, the hypothesis ($2 \in (1, 2]$) is TRUE, but the conclusion ($2 \in (1, 2)$) is FALSE.

10. The implication is TRUE. If x is a number between 1 and 2, then its distance from zero is also between 1 and 2.