ALGEBRA II OBJECTIVES: GR6 and GR7

(GR6) reflection about the x-axis: going from $y = f(x)$ to $y = -f(x)$

(GR6) reflection about the y-axis: going from $y = f(x)$ to $y = f(-x)$

(GR7) absolute value transformation: going from $y = f(x)$ to $y = |f(x)|$

DISCUSSION OF CONCEPT:

There are things that you can DO to an equation of the form $y = f(x)$ that will change the graph in a variety of ways. For example, you can move the graph up or down, left or right, reflect about the x or y axes, stretch or shrink vertically or horizontally.

An understanding of these transformations makes it easy to graph a wide variety of functions, by starting with a 'basic model' and then applying a sequence of transformations to change it to the desired function.

In this discussion, we will explore reflecting about the x-axis and the y-axis, and the absolute value transformation.

When you finish studying this objective, you should be able to do a problem like this:

GRAPH $y = -|\ln(-x)|$:

- Start with the graph of $y = \ln(x)$. (This is the 'basic model'.)
- Replace every x by $-x$, giving the new equation $y = \ln(-x)$.

  This reflects the graph about the y-axis.
- Take the absolute value of the previous y-values, giving the new equation $y = |\ln(-x)|$.

  This takes any part of the graph below the x-axis, and reflects it about the x-axis. Any part of the graph on or above the x-axis remains the same.
- Multiply the previous y-values by -1 , giving the new equation $y = -|\ln(-x)|$.

  This reflects the graph about the x-axis.

Here are ideas that are needed to understand graphical transformations.
First, some ideas regarding functions and the graph of a function:

- A function is a rule: it takes an input, and gives a unique output.
- If x is the input to a function f , then the unique output is called $f(x)$ (which is read as 'f of x').
- The *graph* of a function is a picture of *all* of its (input, output) pairs. We put the inputs along the horizontal axis (the x-axis), and the outputs along the vertical axis (the y-axis).
- Thus, the graph of a function f is a picture of all points of the form $(x, \overbrace{f(x)}^{y\text{-value}})$. Here, x is the input, and $f(x)$ is the corresponding output.
- The equation $y = f(x)$ is an equation in two variables, x and y . A solution is a choice for x and a choice for y that makes the equation true. Of course, in order for this equation to be true, y must equal $f(x)$.

  Thus, solutions to the equation $y = f(x)$ are points of the form $(x, \overbrace{f(x)}^{y\text{-value}})$.
- Compare the previous two ideas! You see that the requests 'graph the function f' and 'graph the equation $y = f(x)$' mean exactly the same thing.

  To "graph the function f" means to show all points of the form $(x, f(x))$.

  To "graph the equation $y = f(x)$" means to show all points of the form $(x, f(x))$.

Ideas regarding reflecting about the x-axis:

- Points on the graph of $y = f(x)$ are of the form $(x, f(x))$.

  Points on the graph of $y = -f(x)$ are of the form $(x, -f(x))$.

  Thus, the graph of $y = -f(x)$ is found by taking the graph of $y = f(x)$ and multiplying the y-values by -1. This reflects the graph about the x-axis.

- **Transformations involving y work the way you would expect them to work—they are intuitive.**

- Here is the thought process you should use when you are given the graph of $y = f(x)$ and asked about the graph of $y = -f(x)$:

  original equation: $y = f(x)$

  new equation: $\overbrace{y}^{\text{the new } y\text{-values}} \ \overbrace{=}^{\text{are}} \ \overbrace{-}^{-1 \text{ times}} \ \overbrace{f(x)}^{\text{the previous } y\text{-values}}$

- In reflection about the x-axis, a point (a, b) on the graph of $y = f(x)$ moves to a point $(a, -b)$ on the graph of $y = -f(x)$.

Ideas regarding reflecting about the y-axis:

- Points on the graph of $y = f(x)$ are of the form $(x, f(x))$.

  Points on the graph of $y = f(-x)$ are of the form $(x, f(-x))$.

- How can we locate these desired points $(x, f(-x))$?

  First, go to the point $(-x, f(-x))$ on the graph of $y = f(x)$.

  This point has the y-value that we want, but it has the wrong x-value.

  The x-value of this point is $-x$, but the desired x-value is just x. Thus, the current x-value must be multiplied by -1; that is, each x-value must be sent to its opposite. The y-value remains the same. This causes the point to reflect about the y-axis. This gives the desired point $(x, f(-x))$.

  Thus, the graph of $y = f(-x)$ is the same as the graph of $y = f(x)$, except that it has been reflected about the y-axis.

- Here is the thought process you should use when you are given the graph of $y = f(x)$ and asked about the graph of $y = f(-x)$:

  original equation: $y = f(x)$

  new equation: $y = f(\ \overbrace{-x}^{\text{replace } x \text{ by } -x} \ )$

  Replacing every x by $-x$ in an equation causes the graph to be reflected about the y-axis.

- In reflection about the y-axis, a point (a, b) on the graph of $y = f(x)$ moves to a point $(-a, b)$ on the graph of $y = f(-x)$.

Ideas regarding the absolute value transformation:

- Points on the graph of $y = f(x)$ are of the form $(x, f(x))$.

  Points on the graph of $y = |f(x)|$ are of the form $(x, |f(x)|)$.

  Thus, the graph of $y = |f(x)|$ is found by taking the graph of $y = f(x)$ and taking the absolute value of the y-values.

  Points with positive y-values stay the same, since the absolute value of a positive number is itself. That is, points above the x-axis don't change.

  Points with $y = 0$ stay the same, since the absolute value of zero is itself. That is, points on the x-axis don't change.

  Points with negative y-values will change, since taking the absolute value of a negative number makes it positive. That is, any point below the x-axis reflects about the x-axis.

  These actions are summarized by saying "any part of the graph below the x-axis flips up".

- Here is the thought process you should use when you are given the graph of $y = f(x)$ and asked about the graph of $y = |f(x)|$:

  original equation: $$y = f(x)$$

  new equation: $$\overbrace{y}^{\text{the new } y\text{-values}} \overbrace{=}^{\text{are}} \overbrace{|f(x)|}^{\text{the absolute value of the previous } y\text{-values}}$$

- In the absolute value transformation, a point (a, b) on the graph of $y = f(x)$ moves to a point $(a, |b|)$ on the graph of $y = |f(x)|$.

Here are some examples that combine reflections about the x-axis and y-axis, and the absolute value transformation.

EXAMPLE:

State the transformations that take the graph of $y = f(x)$ to the graph of $y = -|f(-x)|$.

Equation	Action	Graphical Result
$y = f(x)$	(starting place)	
$y = f(-x)$	replace every x by $-x$	reflect about the y-axis
$y = \|f(-x)\|$	take the absolute value of the previous y-values	any part of the graph below the x-axis flips up
$y = -\|f(-x)\|$	multiply the previous y-values by -1	reflect about the x-axis

EXAMPLE:

State the transformations that take the graph of $y = x^3$ to the graph of $y = -(5 - 3x)^3 + 2$.

Note: This example combines many different transformations.

Note: It is helpful to re-write the function as $y = -(-3x + 5)^3 + 2$.

Note: The sequence of operations given by the expression ' $-3x + 5$ ' is NOT the same as the order in which the transformations are applied. Be careful!

Equation	Action	Graphical Result
$y = x^3$	(starting place)	
$y = (x + 5)^3$	replace every x by $x + 5$	move left 5
$y = (3x + 5)^3$	replace every x by $3x$	horizontal compression, $(a, b) \mapsto (\frac{a}{3}, b)$
$y = (-3x + 5)^3$	replace every x by $-x$	reflect about the y-axis
$y = -(-3x + 5)^3$	multiply the previous y-values by -1	reflect about the x-axis
$y = -(-3x + 5)^3 + 2$	add 2 to the previous y-values	move up 2