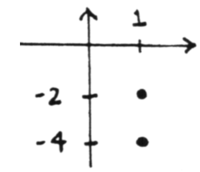
SECTION 7.5 The Area Between Two Curves

IN-SECTION EXERCISES:

EXERCISE 1.

1. Adding the same number to both sides of an inequality yields an equivalent inequality. Adding $-g(x)$ yields the desired result:
$$f(x) \ge g(x) \quad \iff \quad f(x) - g(x) \ge 0$$

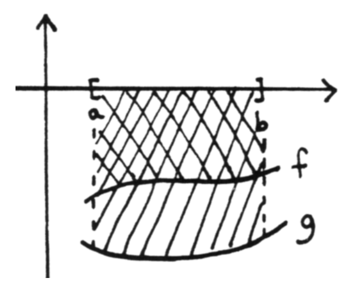
2. The sentence '$f(1) \ge g(1)$' is a true sentence in this case, since:
$$f(1) \ge g(1) \quad \iff \quad -2 \ge -4 \quad \iff \quad 2 \le 4$$
(Remember that multiplying both sides of an inequality by a *negative* number reverses the sense of the inequality.) Since the last inequality '$2 \le 4$' is clearly TRUE, so is the inequality '$f(1) \ge g(1)$'.
In this case:
$$f(1) - g(1) = -2 - (-4) = -2 + 4 = 2$$
Since $f(1) \ge g(1)$, the number $f(1) - g(1)$ gives the distance between these two numbers.



EXERCISE 2.

In this case, both $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ are negative numbers, and the magnitude of $\int_a^b g(x)\,dx$ is larger than the magnitude of $\int_a^b f(x)\,dx$, since there is more area trapped between the x-axis and the graph of g. The numbers $-\int_a^b f(x)\,dx$ and $-\int_a^b g(x)\,dx$ represent the trapped areas. Then:

$$\begin{aligned}\text{desired area} &= \left(-\int_a^b g(x)\,dx\right) - \left(-\int_a^b f(x)\,dx\right)\\ &= -\int_a^b g(x)\,dx + \int_a^b f(x)\,dx\\ &= \int_a^b \big(f(x) - g(x)\big)\,dx\end{aligned}$$

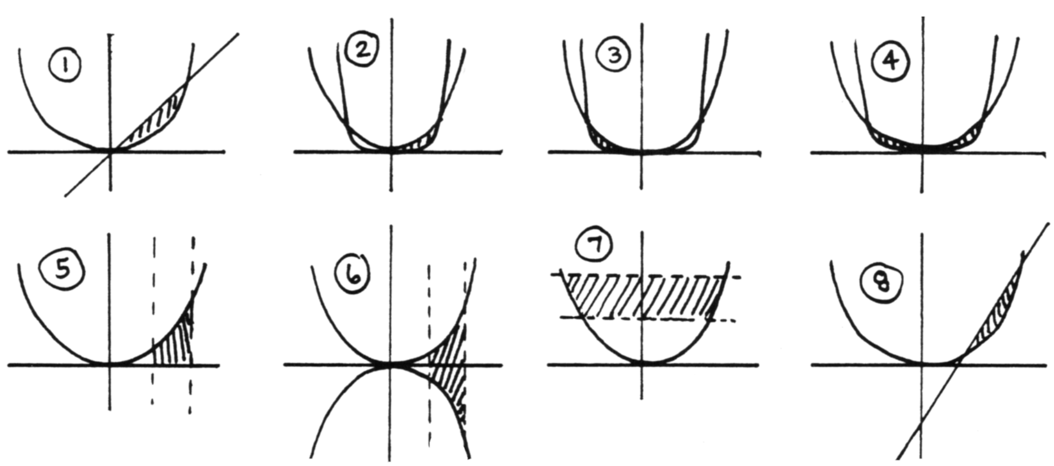


Again, the same formula is obtained.

There are other correct ways to develop the formula in this case.

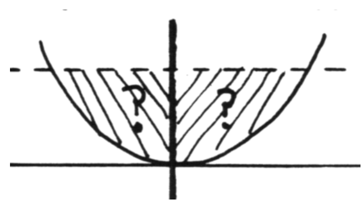
EXERCISE 3.

The desired sketches are given below.



EXERCISE 4.

1. The description, '*the area bounded by* $y = x + 1$ *and* $y = (x-1)^2$' is sufficient, since the area shown is the *only area* having as its boundary only the graphs of these two curves. Of course, to find this area using calculus, the two intersection points need to first be determined.
2. The description, '*the area in the first quadrant bounded by* $y = x^2$, $x = 0$ *and* $y = 2$' correctly describes the given area. Observe that the description '*the area bounded by* $y = x^2$, $x = 0$ *and* $y = 2$' is ambiguous.



EXERCISE 5.

1. The sentence is read as '$x^2 = 4x - 3$ is equivalent to $x = 1$ or $x = 3$'; this means that the sentences '$x^2 = 4x - 3$' and '$x = 1$ or $x = 3$' *always have the same truth values*:
   - whenever $x^2 = 4x - 3$ is true, so is $x = 1$ or $x = 3$
   - whenever $x^2 = 4x - 3$ is false, so is $x = 1$ or $x = 3$
   - whenever $x = 1$ or $x = 3$ is true, so is $x^2 = 4x - 3$
   - whenever $x = 1$ or $x = 3$ is false, so is $x^2 = 4x - 3$
2. A sentence of the form 'A or B' is true when A is true, or B is true, or both A and B are true. That is, at least one of A or B must be true. Thus, the sentence '$x = 1$ or $x = 3$' can be solved by inspection; the only numbers that make it true are 1 and 3. Thus, the only numbers that make $x^2 = 4x - 3$ true are 1 and 3. Check!)
3. If the sentence '$x^2 = 4x - 3$' is false, then so is the sentence '$x = 1$ or $x = 3$'. In this case, therefore, x must be a number different from 1 and 3.
4. A sentence of the form 'A and B' is true only when BOTH A and B are true. Therefore, the mathematical sentence '$x = 1$ and $x = 3$' is false, for every number x. (There are no real numbers which are simultaneously equal to 1 and 3.)

   Choosing, say, $x = 1$, the sentence '$x = 1$ and $x = 3$' becomes '$1 = 1$ and $1 = 3$', which is false. However, the sentence '$x^2 = 4x - 3$' becomes '$1^2 = 4(1) - 3$', which is true. Therefore, the two sentences do NOT always have the same truth values; they are not equivalent.

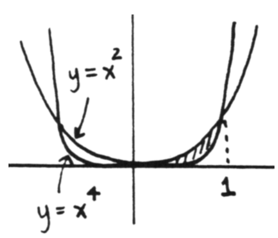
END-OF-SECTION EXERCISES:

1. intersection points:

$$x^2 = x^4 \iff x^4 - x^2 = 0 \iff x^2(x^2 - 1) = 0 \iff (x = 0 \text{ or } x = \pm 1)$$

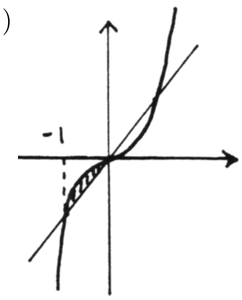
The graph of $y = x^4$ is 'flatter' near zero, so $y = x^2$ is on top. The desired area is:

$$\begin{aligned}\int_0^1 (x^2 - x^4)\,dx &= \left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1 \\ &= \left(\frac{1}{3} - \frac{1}{5}\right) = \frac{5}{15} - \frac{3}{15} = \frac{2}{15}\end{aligned}$$



2. intersection points:

$$x = x^3 \iff x^3 - x = 0 \iff x(x^2-1) = 0 \iff (x = 0 \text{ or } x = \pm 1)$$



The desired area is:

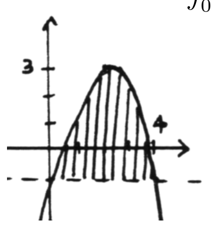
$$\int_{-1}^{0} (x^3 - x)\,dx = \frac{x^4}{4} - \frac{x^2}{2}\,\Big|_{-1}^{0}$$
$$= 0 - (\frac{1}{4} - \frac{1}{2}) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

3. intersection points:

$$-(x-2)^2 + 3 = -1 \iff (x-2)^2 = 4 \iff x - 2 = \pm 2$$
$$\iff x = \pm 2 + 2 \iff (x = 4 \text{ or } x = 0)$$

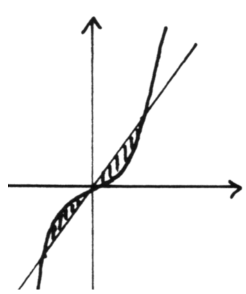
The desired area is:

$$\int_0^4 [-(x-2)^2 + 3 - (-1)]\,dx = \int_0^4 (4 - (x-2)^2)\,dx$$
$$= 4x - \frac{(x-2)^3}{3}\,\Big|_0^4$$
$$= (16 - \frac{8}{3}) - (0 - (-\frac{8}{3})) = 16 - \frac{8}{3} - \frac{8}{3}$$
$$= \frac{48}{3} - \frac{16}{3} = \frac{32}{3}$$

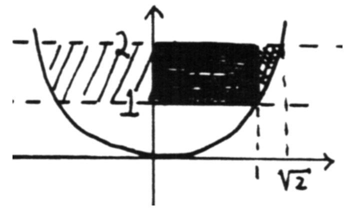


4. The intersection are found (as in problem 2) to be $x = 0$ or $x = \pm 1$. Taking advantage of symmetry, the desired area is:

$$2\int_0^1 (x - x^3)\,dx = 2(\frac{x^2}{2} - \frac{x^4}{4})\,\Big|_0^1 = 2(\frac{1}{2} - \frac{1}{4}) = \frac{1}{2}$$



5. Taking advantage of symmetry, the desired area in the first quadrant is found, and then doubled. The intersection of $y = x^2$ and $y = 1$ is found:



$$x^2 = 1 \quad \iff \quad x = \pm 1$$

The intersection of $y = x^2$ and $y = 2$ is found:

$$x^2 = 2 \quad \iff \quad x = \pm\sqrt{2}$$

The rectangular piece has area $(1-0)(2-1) = 1$. The other piece has area given by:

$$\begin{aligned}\int_1^{\sqrt{2}} (2 - x^2)\,dx &= (2x - \frac{x^3}{3})\,\Big|_1^{\sqrt{2}} \\ &= (2\sqrt{2} - \frac{(\sqrt{2})^3}{3}) - (2 - \frac{1}{3}) \\ &= \frac{6\sqrt{2}}{3} - \frac{2\sqrt{2}}{3} - \frac{5}{3} \\ &= \frac{4\sqrt{2} - 5}{3} \approx 0.219\end{aligned}$$

The total desired area is thus approximately:

$$2(1 + 0.219) = 2.438$$

6. The desired area is:

$$\int_0^2 (x^2 - (-1))\,dx = \int_0^2 (x^2 + 1)\,dx = (\frac{x^3}{3} + x)\,\Big|_0^2 = \frac{8}{3} + 2 = \frac{14}{3}$$

7. intersection point:

$$x^3 = 8 \quad \iff \quad x = 2$$

The desired area is:

$$\int_{-1}^2 (8 - x^3)\,dx = (8x - \frac{x^4}{4})\,\Big|_{-1}^2 = (16 - \frac{16}{4}) - (-8 - \frac{1}{4}) = 12 + 8\frac{1}{4} = 20\frac{1}{4}$$