

SECTION 6.6 Integration by Parts Formula

IN-SECTION EXERCISES:

EXERCISE 2.

$$\frac{d}{dx}(xe^x - e^x) = (xe^x + (1)e^x) - e^x = xe^x$$

EXERCISE 3.

1. The choice for dv must be something that we know how to integrate. Here are the possible choices, with the corresponding choices for u :

$$dv = dx \text{ with corresponding } u = xe^{-x}$$

$$dv = x dx \text{ with corresponding } u = e^{-x}$$

$$dv = e^{-x} dx \text{ with corresponding } u = x$$

2. The only case where $\frac{du}{dx}$ is simpler than u is when $u = x$.

This is how the choices $u = x$ and $dv = e^{-x} dx$ were arrived at.

EXERCISE 4.

1.

$$\begin{aligned} \int x \ln x dx &= (\ln x)\left(\frac{x^2}{2}\right) - \int \frac{x^2}{2} \cdot \frac{1}{x} dx = \frac{1}{2}x^2 \ln x - \frac{1}{2} \int x dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{2} \cdot \frac{x^2}{2} + C = \frac{1}{2}x^2 \left(\ln x - \frac{1}{2}\right) + C \end{aligned}$$

$$\begin{aligned} u &= \ln x & dv &= x dx \\ du &= \frac{1}{x} dx & v &= \frac{x^2}{2} \end{aligned}$$

2.

$$\begin{aligned} \int xe^{3x} dx &= x\left(\frac{1}{3}e^{3x}\right) - \int \frac{1}{3}e^{3x} dx \\ &= \frac{1}{3}xe^{3x} - \frac{1}{3} \cdot \frac{1}{3}e^{3x} + C \\ &= \frac{1}{9}e^{3x}(3x - 1) + C \end{aligned}$$

$$\begin{aligned} u &= x & dv &= e^{3x} dx \\ du &= dx & v &= \frac{1}{3}e^{3x} \end{aligned}$$

3.

$$\begin{aligned} \int x^3 \ln x dx &= (\ln x)\left(\frac{x^4}{4}\right) - \int \frac{x^4}{4} \cdot \frac{1}{x} dx = \frac{1}{4}x^4 \ln x - \frac{1}{4} \int x^3 dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{4} \cdot \frac{x^4}{4} + C = \frac{1}{16}x^4(4 \ln x - 1) + C \end{aligned}$$

$$\begin{aligned} u &= \ln x & dv &= x^3 dx \\ du &= \frac{1}{x} dx & v &= \frac{x^4}{4} \end{aligned}$$

4. $\int \ln 3x dx = (\ln 3x)(x) - \int x \cdot \frac{1}{x} dx = x \ln x - x + C$

$$\begin{aligned} u &= \ln 3x & dv &= dx \\ du &= \frac{1}{3x} \cdot 3 dx & v &= x \\ &= \frac{1}{x} dx \end{aligned}$$

EXERCISE 5.

1.

$$\begin{aligned} \frac{d}{dx}[(x+3) \ln(x+3) - x] &= (x+3) \frac{1}{x+3}(1) + (1) \ln(x+3) - 1 \\ &= 1 + \ln(x+3) - 1 = \ln(x+3) \end{aligned}$$

2.

$$\int 3 \ln(x+1) dx = 3 \int \ln(x+1) dx$$

$$\begin{aligned} \mu &= \ln(x+1) & d\mu &= dx \\ d\mu &= \frac{1}{x+1} dx & \nu &= x+1 \end{aligned}$$

$$\begin{aligned} &= 3 \left[(x+1) \ln(x+1) - \int (x+1) \frac{1}{x+1} dx \right] \\ &= 3 \left[(x+1) \ln(x+1) - x + C \right] \\ &= 3 \left[(x+1) \ln(x+1) - x \right] + K \end{aligned}$$

3.

$$\mu = \ln\left(t - \frac{1}{2}\right) \quad d\mu = dt$$

$$d\mu = \frac{1}{t - \frac{1}{2}} dt \quad \nu = t - \frac{1}{2}$$

$$\int \ln\left(t - \frac{1}{2}\right) dt = \left(t - \frac{1}{2}\right) \ln\left(t - \frac{1}{2}\right) - \int \left(t - \frac{1}{2}\right) \frac{1}{t - \frac{1}{2}} dt$$

$$= \left(t - \frac{1}{2}\right) \ln\left(t - \frac{1}{2}\right) - t + C$$

EXERCISE 6.

2.

$$\mu = x^2 \quad d\mu = 2x dx$$

$$d\mu = 2x dx \quad \nu = \frac{1}{3} e^{3x}$$

$$\int x^2 e^{3x} dx = x^2 \left(\frac{1}{3} e^{3x}\right) - \int \frac{1}{3} e^{3x} (2x) dx$$

$$= \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \int x e^{3x} dx ;$$

$$\mu = x \quad d\mu = dx$$

$$d\mu = dx \quad \nu = \frac{1}{3} e^{3x}$$

$$\int x e^{3x} dx = (x) \left(\frac{1}{3} e^{3x}\right) - \int \frac{1}{3} e^{3x} dx$$

$$= \frac{1}{3} x e^{3x} - \frac{1}{3} \cdot \frac{1}{3} e^{3x} + C$$

$$= \frac{1}{9} e^{3x} (3x - 1) + C$$

Combining results:

$$\int x^2 e^{3x} dx = \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \left[\frac{1}{9} e^{3x} (3x - 1) \right] + C$$

$$= \frac{1}{27} e^{3x} [9x^2 - 2(3x - 1)] + C$$

$$= \frac{1}{27} e^{3x} [9x^2 - 6x + 2] + C$$

EXERCISE 7.

1.

$$\int \frac{x}{(1+x)^6} dx = \int \frac{u-1}{u^6} du = \int (u^{-5} - u^{-6}) du$$

$$\mu = 1+x; \quad x = \mu - 1$$

$$d\mu = dx$$

$$= \frac{u^{-4}}{-4} - \frac{u^{-5}}{-5} + C = -\frac{1}{4(1+x)^4} + \frac{1}{5(1+x)^5} + C$$

Checking:

$$\frac{d}{dx} \left(-\frac{1}{4(1+x)^4} + \frac{1}{5(1+x)^5} \right) = \frac{d}{dx} \left(-\frac{1}{4}(1+x)^{-4} + \frac{1}{5}(1+x)^{-5} \right)$$

$$= -\frac{1}{4} \cdot (-4)(1+x)^{-5} + \frac{1}{5} \cdot (-5)(1+x)^{-6}$$

$$= \frac{1}{(1+x)^5} \cdot \frac{(1+x)}{(1+x)} - \frac{1}{(1+x)^6} = \frac{x}{(1+x)^6}$$

2.

$$\int \frac{x}{(1+x)^6} dx = x \left(\frac{(1+x)^{-5}}{-5} \right) - \int \frac{(1+x)^{-5}}{-5} dx$$

$$= -\frac{x}{5(1+x)^5} + \frac{1}{5} \int (1+x)^{-5} dx$$

$$= -\frac{x}{5(1+x)^5} + \frac{1}{5} \cdot \frac{(1+x)^{-4}}{-4} + C$$

$$= -\frac{x}{5(1+x)^5} - \frac{1}{20(1+x)^4} + C$$

$\mu = x \quad dv = \frac{1}{(1+x)^6} dx$
 $du = dx \quad v = \frac{(1+x)^{-5}}{-5}$

Differentiating:

$$\frac{d}{dx} \left[-\frac{x}{5(1+x)^5} - \frac{1}{20(1+x)^4} \right] = \frac{d}{dx} \left[-\frac{1}{5}x(1+x)^{-5} - \frac{1}{20}(1+x)^{-4} \right]$$

$$= -\frac{1}{5} [x(-5)(1+x)^{-6} + (1)(1+x)^{-5}] - \frac{1}{20}(-4)(1+x)^{-5}$$

$$= \frac{x}{(1+x)^6} - \frac{1}{5(1+x)^5} + \frac{1}{5(1+x)^5}$$

$$= \frac{x}{(1+x)^6}$$

3. Rewriting the solution to (1) as a single fraction with denominator $20(1+x)^5$ yields:

$$-\frac{1}{4(1+x)^4} \cdot \frac{5(1+x)}{5(1+x)} + \frac{1}{5(1+x)^5} \cdot \frac{4}{4} = \frac{-5(1+x) + 4}{20(1+x)^5} = \frac{-5x - 1}{20(1+x)^5}$$

Doing the same with (2):

$$-\frac{x}{5(1+x)^5} \cdot \frac{4}{4} - \frac{1}{20(1+x)^4} \cdot \frac{(1+x)}{(1+x)} = \frac{-4x - 1 - x}{20(1+x)^5}$$

$$= \frac{-5x - 1}{20(1+x)^5}$$

Compare!

END-OF-SECTION EXERCISES:

1.

$$\int (e^x - 1)^2 dx = \int (e^{2x} - 2e^x + 1) dx$$

$$= \frac{1}{2}e^{2x} - 2e^x + x + C$$

2.

$$\int \frac{\ln(x^2 + 2x + 1)}{x + 1} dx = \int \frac{\ln(x + 1)^2}{x + 1} dx$$

$$= 2 \int \frac{\ln(x + 1)}{x + 1} dx$$

$$= 2 \int u du = 2 \frac{u^2}{2} + C = (\ln(x + 1))^2 + C$$

$u = \ln(x + 1)$
 $du = \frac{1}{x + 1} dx$

3.

$$u = 1 + e^x$$

$$du = e^x dx$$

$$\int \frac{e^x}{1 + e^x} dx = \int \frac{1}{u} du$$

$$= \ln |u| + C = \ln |1 + e^x| + C$$

4.

$$\int \ln \frac{1+x}{x} dx = \int \ln(1+x) - \ln x dx$$

$$= [(x+1)\ln(x+1) - x] - [x \ln x - x] + C$$

$$= (x+1)\ln(x+1) - x \ln x + C$$

(PARTS,
TWICE!)

5.

$$\int \sqrt{\frac{e^t}{2}} dt = \frac{1}{\sqrt{2}} \int (e^t)^{1/2} dt = \frac{1}{\sqrt{2}} \int e^{t/2} dt$$

$$= \frac{1}{\sqrt{2}} \cdot 2e^{t/2} + C = \sqrt{2}(e^t)^{1/2} + C$$

$$= \sqrt{2e^t} + C$$

6.

$$\int \frac{x}{\sqrt{x^4 \ln x}} dx = \int \frac{x}{x^2 \sqrt{\ln x}} dx = \int \frac{1}{x \sqrt{\ln x}} dx$$

$$= \int \frac{1}{\sqrt{u}} du = \int u^{-1/2} du = 2u^{1/2} + C$$

$$= 2\sqrt{\ln x} + C$$

$$u = \ln x$$

$$du = \frac{1}{x} dx$$