SECTION 6.5 More on Substitution

IN-SECTION EXERCISES:

EXERCISE 1.

'Role Reversal':

$$\int \frac{3t}{t-1}\, dt = 3 \int \frac{t}{t-1}\, dt$$

$$= 3 \int \frac{u+1}{u}\, du = 3 \int 1 + \frac{1}{u}\, du$$

$$= 3(u + \ln|u|) + C = 3u + 3\ln|u| + C$$

$$= 3(t-1) + 3\ln|t-1| + C$$

$$= 3t + 3\ln|t-1| + K$$

$u = t-1 \; ; \; t = u+1$

$du = dt$

Using long division:

$$\int \frac{3t}{t-1}\, dt = 3 \int \frac{t}{t-1}\, dt$$

$$= 3 \int 1 + \frac{1}{t-1}\, dt = 3(t + \ln|t-1|) + C$$

$t-1\,\overline{\big)\,t}$
$\quad\,-(t-1)$
$\quad\quad\quad 1$

EXERCISE 2.

'Role Reversal':

$$\int \frac{2t}{3t-1}\, dt = 2 \int \frac{t}{3t-1}\, dt = 2 \int \frac{\frac{u+1}{3}}{u} \cdot \frac{du}{3}$$

$$= \frac{2}{9} \int \frac{u+1}{u}\, du = \frac{2}{9} \int 1 + \frac{1}{u}\, du$$

$$= \frac{2}{9}(u + \ln|u|) + C$$

$$= \frac{2}{9}(3t - 1 + \ln|3t-1|) + C$$

$$= \frac{2}{9}(3t + \ln|3t-1|) + K$$

$u = 3t - 1 \; ; \; t = \dfrac{u+1}{3}$

$du = 3dt \; ; \; dt = \dfrac{du}{3}$

Using long division:

$$\int \frac{2t}{3t-1}\, dt = 2 \int \frac{t}{3t-1}\, dt$$

$$= 2 \int \frac{1}{3} + \frac{1/3}{3t-1}\, dt$$

$$= 2 \int \frac{1}{3}\, dt + \frac{2}{3} \int \frac{1}{3t-1}\, dt$$

$$= \frac{2}{3}t + \frac{2}{9} \int \frac{3}{3t-1}\, dt$$

$$= \frac{2}{3}t + \frac{2}{9} \int \frac{1}{u}\, du$$

$$= \frac{2}{3}t + \frac{2}{9} \ln|3t-1| + C$$

$\dfrac{1}{3}$

$3t-1\,\overline{\big)\,t}$
$\quad\,-(t-\tfrac{1}{3})$
$\quad\quad\quad \tfrac{1}{3}$

$u = 3t - 1$

$du = 3 dt$

Compare! As you get more proficient in integrating, you will probably be able to leave out a number of the steps included above.

EXERCISE 3.

Eventually, you should be above to solve these problems with the number of steps given here:

1.

$u = t+1; \quad t = u-1$
$du = dt$

$$\int t(t+1)^7 \, dt = \int (u-1)u^7 \, du = \int u^8 - u^7 \, du$$
$$= \frac{(t+1)^9}{9} - \frac{(t+1)^8}{8} + C$$

2.

$$\int \frac{5x}{\sqrt{(3-2x)^3}} \, dx = 5 \int \frac{\frac{3-u}{2}}{u^{3/2}} \cdot \frac{du}{-2}$$

$u = 3-2x;$

$x = \frac{u-3}{-2} = \frac{3-u}{2}$

$du = -2dx;$

$dx = \frac{du}{-2}$

$$= -\frac{5}{4} \int (3-u)u^{-3/2} \, du$$
$$= -\frac{5}{4} \int 3u^{-3/2} - u^{-1/2} \, du$$
$$= -\frac{5}{4}(3 \cdot \frac{u^{-1/2}}{-1/2} - \frac{u^{1/2}}{1/2}) + C$$
$$= -\frac{5}{4}(\frac{-6}{\sqrt{3-2x}} - 2\sqrt{3-2x}) + C$$

3.

$w = u^2 + 1$
$dw = 2u\,du$

$$\int u\sqrt{u^2+1} \, du = \frac{1}{2} \int \sqrt{w} \, dw$$
$$= \frac{1}{2} \cdot \frac{2}{3}w^{3/2} + C$$
$$= \frac{1}{3}\sqrt{(u^2+1)^3} + C$$

EXERCISE 4.

2.

$$\frac{d}{dx}(2\sqrt{x} - 2\ln|1+\sqrt{x}|) = 2 \cdot \frac{1}{2}x^{-1/2} - 2 \cdot \frac{1}{1+\sqrt{x}} \cdot \frac{1}{2}x^{-1/2}$$
$$= \frac{1}{\sqrt{x}} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}} - \frac{1}{\sqrt{x}(1+\sqrt{x})}$$
$$= \frac{1+\sqrt{x}-1}{\sqrt{x}(1+\sqrt{x})}$$
$$= \frac{1}{1+\sqrt{x}}$$

EXERCISE 5.

'Role Reversal':

$$\int \frac{x}{\sqrt{x-1}} \, dx = \int \frac{u+1}{\sqrt{u}} \, du$$

$u = x-1; \quad x = u+1$

$du = dx$

$$= \int u^{1/2} + u^{-1/2} \, du$$
$$= \frac{2}{3}u^{3/2} + 2u^{1/2} + C$$
$$= \frac{2}{3}\sqrt{(x-1)^3} + 2\sqrt{x-1} + C$$

Rationalizing Substitution:

$$\int \frac{x}{\sqrt{x-1}}\, dx = \int \frac{u^2+1}{u}\cdot 2u\, du$$

$u = \sqrt{x-1}$;

$u^2 = x - 1$; $x = u^2 + 1$

$2u\frac{du}{dx} = 1$; $2u\, du = dx$

$$= 2\int u^2 + 1\, du = 2(\frac{u^3}{3}+u)+C$$

$$= \frac{2}{3}(\sqrt{x-1})^3 + 2\sqrt{x-1} + C$$

Compare!

END-OF-SECTION EXERCISES:

1.
$$\int \frac{e^{2x}+1}{5}\, dx = \frac{1}{5}\int e^{2x}+1\, dx = \frac{1}{5}(\frac{1}{2}e^{2x}+x)+C$$

2.
$$\int xe^{(3x^2-1)}\, dx = \frac{1}{6}\int e^{3x^2-1}\, 6x\, dx$$

$u = 3x^2 - 1$

$du = 6x\, dx$

$$= \frac{1}{6}\int e^u\, du = \frac{1}{6}e^u + C$$

$$= \frac{1}{6}e^{3x^2-1}+C$$

3.
$$\int \frac{t}{\sqrt[3]{4t^2-1}}\, dt = \int t(4t^2-1)^{-1/3}\, dt$$

$u = 4t^2 - 1$

$du = 8t\, dt$

$$= \frac{1}{8}\int (4t^2-1)^{-1/3}\, 8t\, dt = \frac{1}{8}\int u^{-1/3}\, du$$

$$= \frac{1}{8}\cdot \frac{3}{2}u^{2/3}+C = \frac{3}{16}(4t^2-1)^{2/3}+C$$

$$= \frac{3}{16}\sqrt[3]{(4t^2-1)^2}+C$$

4.
$$\int \frac{x}{2x-1}\, dx = \int \frac{\frac{u+1}{2}}{u}\cdot \frac{du}{2}$$

$u = 2x - 1$; $x = \frac{u+1}{2}$

$du = 2dx$; $dx = \frac{du}{2}$

$$= \frac{1}{4}\int \frac{u+1}{u}\, du = \frac{1}{4}\int 1 + \frac{1}{u}\, du$$

$$= \frac{1}{4}(u + \ln|u|)+C = \frac{1}{4}(2x-1+\ln|2x-1|)+C$$

$$= \frac{1}{4}(2x+\ln|2x-1|)+K$$

5.
$$\int x(x+1)^3(x-1)^3\, dx = \int x[(x+1)(x-1)]^3\, dx$$

$u = x^2 - 1$

$du = 2x\, dx$

$$= \int x(x^2-1)^3\, dx = \frac{1}{2}\int (x^2-1)^3\, 2x\, dx$$

$$= \frac{1}{2}\int u^3\, du = \frac{1}{2}\cdot \frac{u^4}{4}+C$$

$$= \frac{(x^2-1)^4}{8}+C$$

6.

$$\int \frac{2t-1}{t}\,dt = \int 2 - \frac{1}{t}\,dt$$
$$= 2t - \ln|t| + C$$

7.

$$\int \frac{(\ln x)^3}{3x}\,dx = \frac{1}{3}\int (\ln x)^3 \cdot \frac{1}{x}\,dx$$
$$= \frac{1}{3}\int u^3\,du = \frac{1}{3}\cdot\frac{u^4}{4} + C$$

$$u = \ln x$$
$$du = \frac{1}{x}\,dx$$

$$= \frac{(\ln x)^4}{12} + C$$