SECTION 6.4 The Substitution Technique for Integration

IN-SECTION EXERCISES:

EXERCISE 1.

$$\begin{aligned}\frac{d}{dx}\left(\frac{(3-4x^2)^{101}}{101}\right) &= \frac{1}{101}\cdot 101(3-4x^2)^{101-1}(-8x)\\ &= (3-4x^2)^{100}(-8x)\end{aligned}$$

EXERCISE 2.

$$\begin{aligned}\int \overbrace{(3-4x^2)}^{u}{}^{100}\,\overbrace{(-8x)\,dx}^{du} &= \int u^{100}\,du && \text{(rewrite in terms of } u\text{)}\\ &= \frac{u^{101}}{101} + C && \text{(integrate the `new' problem)}\\ &= \frac{(3-4x^2)^{101}}{101} + C && \text{(transform back to } x\text{)}\end{aligned}$$

$u = 3 - 4x^2$

$du = -8x\ dx$

EXERCISE 3.

1.

$$\begin{aligned}\frac{d}{dx}\left(-\frac{1}{8}\cdot\frac{(3-4x^2)^{101}}{101}\right) &= -\frac{1}{8}\cdot\frac{1}{101}(101)(3-4x^2)^{100}(-8x)\\ &= (3-4x^2)^{100}x\end{aligned}$$

2. Linearity was used in going from

$$\int (3-4x^2)^{100}\left(\frac{-8}{-8}\right)x\,dx \qquad \text{to} \qquad \frac{1}{-8}\int \overbrace{(3-4x^2)}^{u}{}^{100}\,\overbrace{(-8x)\,dx}^{du}$$

EXERCISE 4.

1. $\frac{du}{dx} = 3x^2$; it was noted that the variable part of this derivative, x^2, also appeared as a factor in the integrand

2.

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^3-1}}\,dx &= \frac{1}{3}\int \frac{3x^2}{\sqrt{x^3-1}}\,dx && \text{(multiply by 1 in form } \tfrac{3}{3}\text{; linearity)}\\ &= \frac{1}{3}\int \frac{1}{\sqrt{u}}\,du && \text{(rewrite in terms of } u\text{)}\\ &= \frac{1}{3}\int u^{-1/2}\,du && \text{(rewrite with fractional exponents)}\\ &= \frac{1}{3}\cdot\frac{u^{1/2}}{1/2} + C && \text{(simple power rule)}\\ &= \frac{2}{3}\sqrt{x^3-1} + C && \text{(rewrite in terms of } x\text{)}\end{aligned}$$

$u = x^3 - 1$

$du = 3x^2\ dx$

3.

$$\begin{aligned}\frac{d}{dx}\left(\frac{2}{3}\sqrt{x^3-1}\right) &= \frac{d}{dx}\left(\frac{2}{3}(x^3-1)^{1/2}\right)\\ &= \frac{2}{3}\cdot\frac{1}{2}(x^3-1)^{-1/2}(3x^2)\\ &= \frac{x^2}{\sqrt{x^3-1}}\end{aligned}$$

EXERCISE 5.

1. The derivative of y^2+2y+1 is $2y+2 = 2(y+1)$; it was noted that this derivative appeared (off only by a constant) as a factor in the integrand.

2.

$$\begin{aligned}\int \frac{x+1}{(x^2+2x+1)^3}\,dx &= \int \frac{(\frac{1}{2})(2)(x+1)}{(x^2+2x+1)^3}\,dx = \frac{1}{2}\int \frac{2x+2}{(x^2+2x+1)^3}\,dx\\ &= \frac{1}{2}\int \frac{1}{u^3}\,du = \frac{1}{2}\int u^{-3}\,du\\ &= \frac{1}{2}\cdot\frac{u^{-2}}{-2} + C = -\frac{1}{4u^2} + C\\ &= -\frac{1}{4(x^2+2x+1)^2} + C\end{aligned}$$

$u = x^2 + 2x + 1$
$du = (2x + 2)\ dx$

3.

$$\begin{aligned}\frac{d}{dy}\left(-\frac{1}{4(y^2+2y+1)^2}\right) &= \frac{d}{dy}\left(-\frac{1}{4}(y^2+2y+1)^{-2}\right)\\ &= -\frac{1}{4}(-2)(y^2+2y+1)^{-3}(2y+2)\\ &= \frac{1}{2}(y^2+2y+1)^{-3}2(y+1)\\ &= \frac{y+1}{(y^2+2y+1)^3}\end{aligned}$$

EXERCISE 6.

1. Define $g(x) = \ln|x|$, so that $g(f(x)) = \ln|f(x)|$.
   Recall that $g'(x) = \frac{1}{x}$, for all $x \neq 0$. By the Chain Rule:

$$\frac{d}{dx}\ln|f(x)| = \frac{d}{dx}g(f(x)) = g'(f(x))\cdot f'(x) = \frac{1}{f(x)}\cdot f'(x)$$

2.

$$f(0) = \frac{\ln|3\cdot 0+5| + 3 - \ln 5}{3} = \frac{3}{3} = 1$$

So, the point $(0,1)$ lies on the graph of f. Also:

$$f'(x) = \frac{d}{dx}\left(\frac{1}{3}(\ln|3x+5| + 3 - \ln 5)\right) = \frac{1}{3}(\frac{1}{3x+5}(3) + 0) = \frac{1}{3x+5}$$

END-OF-SECTION EXERCISES:

1.

$$\int (2x-1)^{17}\,dx = \int (2x-1)^{17}\frac{2}{2}\,dx = \frac{1}{2}\int (2x-1)^{17}(2\,dx)$$

$u = 2x - 1$
$du = 2\,dx$

$$= \frac{1}{2}\int u^{17}\,du = \frac{1}{2}\frac{u^{18}}{18} + C$$
$$= \frac{1}{36}(2x-1)^{18} + C$$

2.

$$\int 5t\sqrt{t^2+3}\,dt = 5\int (t^2+3)^{1/2}\frac{2}{2}t\,dt = \frac{5}{2}\int (t^2+3)^{1/2}(2t\,dt)$$

$u = t^2 + 3$
$du = 2t\,dt$

$$= \frac{5}{2}\int u^{1/2}\,du = \frac{5}{2}\frac{u^{3/2}}{3/2} + C$$
$$= \frac{5}{3}(t^2+3)^{3/2} + C = \frac{5}{3}\sqrt{(t^2+3)^3} + C$$

3.

$u = \ln 4x$
$du = \frac{1}{4x}\cdot 4\,dx$
$= \frac{1}{x}\,dx$

$$\int \frac{3\ln 4x}{x}\,dx = 3\int \ln 4x(\frac{1}{x}\,dx) = 3\int u\,du$$
$$= 3\frac{u^2}{2} + C = \frac{3}{2}(\ln 4x)^2 + C$$

4.

$$\int (4e^{2t} + e^{1+t})\,dt = 4\int e^{2t}\,dt + \int e^{1+t}\,dt$$

$u = 1+t$
$du = dt$

$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$

$$= 4\cdot\frac{1}{2}e^{2t} + \int e^u\,du$$
$$= 2e^{2t} + e^{1+t} + C$$

5.

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\frac{2}{2}\,dx = 2\int e^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}\,dx$$

$u = \sqrt{x} = x^{1/2}$
$du = \frac{1}{2}x^{-1/2}\,dx$
$= \frac{1}{2\sqrt{x}}\,dx$

$$= 2\int e^u\,du = 2e^u + C$$
$$= 2e^{\sqrt{x}} + C$$

Try re-doing this problem, taking $u = e^{\sqrt{x}}$.

6.

$$\int \frac{-1}{2u+5}\,du = -\int \frac{1}{2u+5}\cdot\frac{2}{2}\,du = -\frac{1}{2}\int \frac{1}{2u+5}(2\,du)$$
$$= -\frac{1}{2}\int \frac{1}{w}\,dw = -\frac{1}{2}\ln|w| + C$$

$w = 2u + 5$
$dw = 2du$

$$= -\frac{1}{2}\ln|2u+5| + C$$

7.

$$\int \frac{4t+2}{\sqrt{(t^2+t+1)^3}}\,dt = \int \frac{2(2t+1)}{(t^2+t+1)^{3/2}}\,dt = 2\int (t^2+t+1)^{-3/2}(2t+1)\,dt$$

$u = t^2 + t + 1$
$du = (2t+1)\,dt$

$$= 2\int u^{-3/2}\,du = 2\cdot\frac{u^{-1/2}}{-1/2} + C$$

$$= -4(t^2+t+1)^{-1/2} + C = \frac{-4}{\sqrt{t^2+t+1}} + C$$

8.

$u = e^x + 1$
$du = e^x\,dx$

$$\int (e^x+1)^5 \cdot 3e^x\,dx = 3\int (e^x+1)^5 e^x\,dx = 3\int u^5\,du$$

$$= 3\frac{u^6}{6} + C = \frac{1}{2}(e^x+1)^6 + C$$

9. First, find *all* functions f with the specified derivative:

$$f(x) = \int e^x(e^x+1)^3\,dx = \int u^3\,du$$

$u = e^x + 1$
$du = e^x\,dx$

$$= \frac{u^4}{4} + C$$

$$= \frac{(e^x+1)^4}{4} + C$$

Then, for the graph of f to pass through the point $(0,4)$, it must be that $f(0) = 4$:

$$f(0) = 4 \iff \frac{(e^0+1)^4}{4} + C = 4$$
$$\iff 4 + C = 4 \iff C = 0$$

Take: $f(x) = \dfrac{(e^x+1)^4}{4}$

10. Since $d'(t) = v(t)$, the distance function is an antiderivative of the velocity function. First, find *all* antiderivatives:

$$d(t) = \int (t-2)^3\,dt = \int u^3\,du = \frac{u^4}{4} + C = \frac{(t-2)^4}{4} + C$$

$u = t - 2$
$du = dt$

Then, knowing that $d(1) = \frac{1}{2}$ yields:

$$\frac{(1-2)^4}{4} + C = \frac{1}{2} \iff C = \frac{1}{4}$$

Take: $d(t) = \dfrac{(t-2)^4}{4} + \dfrac{1}{4}$

11. a) The student pulled a *variable* out of the integral in going from

$$\int \frac{2x}{2x}(x^2+1)^5\,dx \quad \text{TO} \quad \frac{1}{2x}\int (x^2+1)^5(2x\,dx)\ ,$$

which is NOT allowed.

b) The student's 'solution' is NOT correct:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{(x^2+1)^6}{12x}\right) &= \frac{(12x)6(x^2+1)^5(2x)-(x^2+1)^6(12)}{(12x)^2} \\
&= \frac{12(x^2+1)^5[12x^2-(x^2+1)]}{144x^2} \\
&= \frac{(x^2+1)^5(11x^2-1)}{12x^2}\ ,
\end{aligned}$$

which is NOT equal to $(x^2+1)^5$. (For example, substitute $x = 1$ into both formulas, to see that they are NOT the same.)