SECTION 5.1 Increasing and Decreasing Functions

IN-SECTION EXERCISES:

EXERCISE 1.

It is assumed that the pattern illustrated at the graph boundaries continues. Some approximation is necessary.

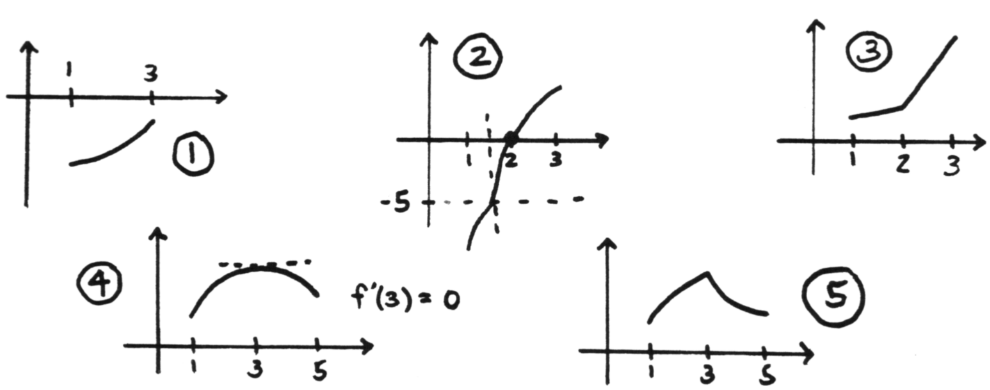
f increases on $(-\infty,-3)\cup(0,2)\cup(4,6)$

f decreases on $(-3,0)\cup(6,10)\cup(10,\infty)$

f neither increases nor decreases on $(2,4)$

EXERCISE 2.

There are many possible correct answers.



EXERCISE 3.

1. Read this as 'x_1 less than x_2 implies $f(x_1)$ is less than or equal to $f(x_2)$'.

   Alternately, read as: 'If x_1 is less than x_2, then $f(x_1)$ is less than or equal to $f(x_2)$'.

   The hypothesis is the sentence '$x_1 < x_2$'; the conclusion is '$f(x_1) \le f(x_2)$'.
2. The hypothesis '$x_1 < x_2$' becomes '$1 < 3$', which is true.

   The conclusion '$f(x_1) \le f(x_2)$' becomes '$-1 \le -0.5$', which is true.

   When both the hypothesis and conclusion of an implication are true, then the implication is true. Thus, the sentence

   $$\overbrace{\underbrace{1<3}_{\text{true}} \implies \underbrace{-1 \le -0.5}_{\text{true}}}^{\text{true}}$$

   is true.
3. The hypothesis is still '$1 < 3$', which is true.

   The conclusion is '$-0.5 \le -1$', which is false.

   When the hypothesis is true, but the conclusion is false, the implication is false. Thus, the sentence

   $$\overbrace{\underbrace{1<3}_{\text{true}} \implies \underbrace{-0.5 \le -1}_{\text{false}}}^{\text{false}}$$

   is false.

4. The sentence

$$\text{for all } x_1, x_2 \in I,\ \ x_1 < x_2 \implies f(x_1) \le f(x_2) \qquad (*)$$

is FALSE, if there is at least one choice for x_1 and x_2 from the interval I that makes '$x_1 < x_2 \implies f(x_1) \le f(x_2)$' false.

Plot the three known points: choosing x_1 to be 2, and x_2 to be 3, then the hypothesis '$2 < 3$' is true, but the conclusion '$2 \le 1$' is false. Thus, the sentence (*) is false.

5. No matter what choices of x_1 and x_2 are made from the three given points, the sentence

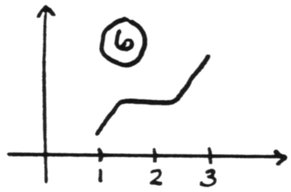
$$x_1 < x_2 \implies f(x_1) \le f(x_2)$$

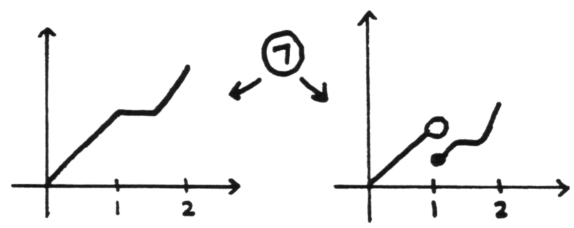
is true. (Remember: if the hypothesis of an implication is false, then the sentence is automatically (vacuously) true.) HOWEVER, THERE ARE MANY OTHER POSSIBLE CHOICES FROM THE INTERVAL I THAT WE DO NOT KNOW ABOUT. Thus, without additional information about f, the truth of the sentence

$$\text{for all } x_1, x_2 \in I,\ \ x_1 < x_2 \implies f(x_1) \le f(x_2)$$

cannot be decided.

6.

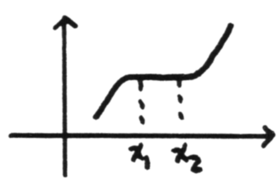




7. If f is continuous at $x = 1$, increases on $(0, 1)$ and is nondecreasing on $(1, 2)$, then it must be nondecreasing on $(0, 2)$. However, if f is not continuous at $x = 1$, then f may not be nondecreasing on $(0, 2)$. Both cases are illustrated above.

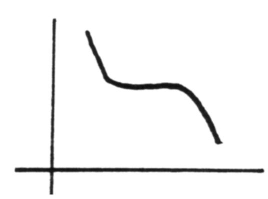
8. TRUE. Whenever the sentence '$f(x_1) < f(x_2)$' is true, so is the sentence '$f(x_1) \le f(x_2)$'.

FALSE. The sentence '$f(x_1) \le f(x_2)$' can be true, without having the sentence '$f(x_1) < f(x_2)$' true, as illustrated below.



9. DEFINITION. A function f is *nonincreasing* on an interval I if and only if:

$$\text{for all } x_1, x_2 \in I,\ \ x_1 < x_2 \implies f(x_1) \ge f(x_2)$$



EXERCISE 4.

Proof. Let f be differentiable on (a, b) and suppose that $f'(x) < 0\ \ \forall\, x \in (a, b)$. Choose *any* x_1, x_2 in (a, b) with $x_1 < x_2$ (so that $x_2 - x_1 > 0$). Observe that x_1 cannot be a, since $a \notin (a, b)$. Similarly, x_2 cannot be b.

Since f is differentiable at x_1 and x_2 (by hypothesis), f must also be continuous at x_1 and x_2. Thus, f is not only differentiable on the open interval (x_1, x_2), but also continuous on the closed interval $[x_1, x_2]$. Thus, the Mean Value Theorem guarantees existence of a number c in (x_1, x_2) for which:

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But since $c \in (a,b)$ and $f'(x) < 0 \;\; \forall\, x \in (a,b)$, we have $f'(c) < 0$. Thus:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0$$

Multiplying both sides of this inequality by the positive number $x_2 - x_1$ yields the equivalent inequality

$$f(x_2) - f(x_1) < 0 \ ,$$

that is, $f(x_2) < f(x_1)$. It has been shown that *whenever* $x_1, x_2 \in I$ satisfy $x_1 < x_2$, it is also true that $f(x_1) > f(x_2)$. So, f is decreasing on I. ∎

EXERCISE 5.

Proof. Let f be differentiable on (a,b) with $f'(x) \ge 0 \;\; \forall\, x \in (a,b)$. Choose any $x_1, x_2 \in (a,b)$ with $x_1 < x_2$. Since f is differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$, the MVT guarantees existence of a number $c \in (x_1, x_2)$ for which:

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But $f'(c) \ge 0$ yields the desired conclusion that $x_1 < x_2 \implies f(x_1) \le f(x_2)$. ∎

EXERCISE 6.

1. The product ab is positive if both factors are positive, or if both factors are negative.
2. If $a = 1$ and $b = 2$, the sentence '$ab > 0$' becomes '$1 \cdot 2 > 0$', which is true. The sentence

$$(a > 0 \text{ and } b > 0) \text{ or } (a < 0 \text{ and } b < 0)$$

becomes the true sentence:

$$\overbrace{\overbrace{(\overbrace{1 > 0}^{\text{true}} \text{ and } \overbrace{2 > 0}^{\text{true}})}^{\text{true}} \text{ or } \overbrace{(\overbrace{1 < 0}^{\text{false}} \text{ and } \overbrace{2 < 0}^{\text{false}})}^{\text{false}}}^{\text{true}}$$

3. If $a = 1$ and $b = -2$, the sentence '$ab > 0$' becomes '$1 \cdot (-2) > 0$', which is false. The sentence

$$(a > 0 \text{ and } b > 0) \text{ or } (a < 0 \text{ and } b < 0)$$

becomes the false sentence:

$$\overbrace{\overbrace{(\overbrace{1 > 0}^{\text{true}} \text{ and } \overbrace{-2 > 0}^{\text{false}})}^{\text{false}} \text{ or } \overbrace{(\overbrace{1 < 0}^{\text{false}} \text{ and } \overbrace{-2 < 0}^{\text{true}})}^{\text{false}}}^{\text{false}}$$

EXERCISE 7.

1. The number 6 is always positive, and hence does not contribute to the sign of $P'(x)$. Only the *variable* factors need be considered when finding the sign of $P'(x)$.
2. The fact was used to replace the set

$$\{x \mid x > 1 \text{ and } x > -2\}$$

with the set:

$$\{x \mid x > 1\}$$

In order for the sentence '$x > 1$ and $x > -2$' to be true, BOTH component sentences must be true. This happens only when x is greater than 1. Thus, the sentences '$x > 1$ and $x > -2$' and '$x > 1$' have precisely the same solution sets.

EXERCISE 8.

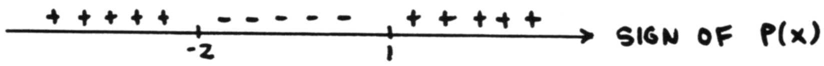
Here's a correct solution, phrased in terms of *equivalence of sentences*, rather than *equality of sets*:

$$\begin{aligned} P'(x) < 0 \quad &\iff \quad 6x^2 + 6x - 12 < 0 \\ &\iff \quad 6(x-1)(x+2) < 0 \\ &\iff \quad (x - 1 < 0 \text{ and } x + 2 > 0) \text{ or } (x - 1 > 0 \text{ and } x + 2 < 0) \\ &\iff \quad (x < 1 \text{ and } x > -2) \text{ or } (x > 1 \text{ and } x < -2) \end{aligned}$$

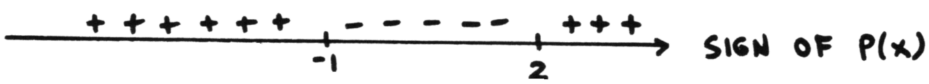
Observe that '$x > 1$ and $x < -2$' is always false. The only time that '$x < 1$ and $x > -2$' is true is for $x \in (-2, 1)$. Thus, P decreases on $(-2, 1)$.

END-OF-SECTION EXERCISES:

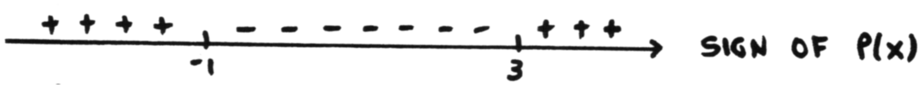
1. $P(x) = (x+2)(x-1)$; $P(x) = 0 \iff (x = -2 \text{ or } x = 1)$
   Test Points: $P(-3) = (-)(-) > 0$; $P(0) = (+)(-) < 0$; $P(2) = (+)(+) > 0$



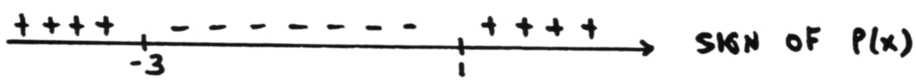
2. $P(x) = (x-2)(x+1)$; $P(x) = 0 \iff (x = 2 \text{ or } x = -1)$
   Test Points: $P(-2) = (-)(-) > 0$; $P(0) = (-)(+) < 0$; $P(3) = (+)(+) > 0$



3. $P(x) = 2(x^2 - 2x - 3) = 2(x-3)(x+1)$; $P(x) = 0 \iff (x = 3 \text{ or } x = -1)$
   Test Points: $P(-2) = (-)(-) > 0$; $P(0) = (-)(+) < 0$; $P(4) = (+)(+) > 0$



4. $P(x) = 3(x^2 + 2x - 3) = 3(x+3)(x-1)$; $P(x) = 0 \iff (x = -3 \text{ or } x = 1)$
   Test Points: $P(-4) = (-)(-) > 0$; $P(0) = (+)(-) < 0$; $P(2) = (+)(+) > 0$



5. Try to find a pair of integers with product $(12)(3) = 36$ and sum -13; use -4 and -9. Then:

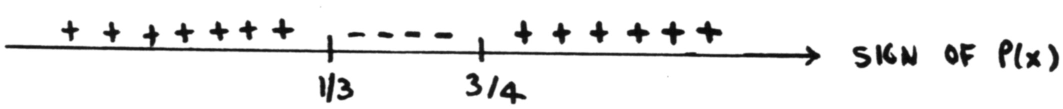
$$\begin{aligned} 12x^2 - 13x + 3 &= 12x^2 - 4x - 9x + 3 \\ &= 4x(3x-1) - 3(3x-1) \\ &= (4x-3)(3x-1) \end{aligned}$$

(THIS TECHNIQUE FOR FACTORING IS DISCUSSED IN DETAIL IN SECTION 5.5)

   Then:

$$P(x) = 0 \quad \iff \quad (4x - 3 = 0 \text{ or } 3x - 1 = 0) \quad \iff \quad x = \frac{3}{4} \text{ or } x = \frac{1}{3}$$

   Test Points: $P(0) = (-)(-) > 0$; $P(\frac{1}{2}) = (-)(+) < 0$; $P(1) = (+)(+) > 0$



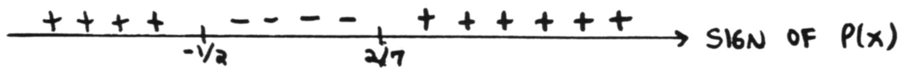
6. Try to find a pair of integers with product $(14)(-2) = -28$ and sum 3; use 7 and -4. Then:

$$\begin{aligned} 14x^2 + 3x - 2 &= 14x^2 + 7x - 4x - 2 \\ &= 7x(2x+1) - 2(2x+1) \\ &= (7x-2)(2x+1) \end{aligned}$$

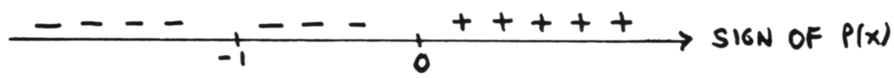
Then:

$$P(x) = 0 \iff (7x - 2 = 0 \text{ or } 2x + 1 = 0) \iff x = \frac{2}{7} \text{ or } x = -\frac{1}{2}$$

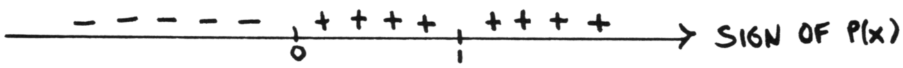
Test Points: $P(-1) = (-)(-) > 0$; $P(0) = (-)(+) < 0$; $P(1) = (+)(+) > 0$



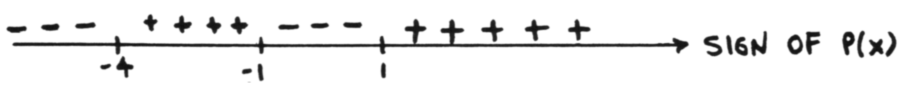
7. $P(x) = x(x^2 + 2x + 1) = x(x+1)^2$; $P(x) = 0 \iff (x = 0 \text{ or } x = -1)$
Test Points: $P(-2) = (-)(+) < 0$; $P(-\frac{1}{2}) = (-)(+) < 0$; $P(1) = (+)(+) > 0$



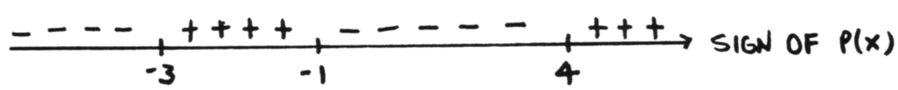
8. $P(x) = x(x^2 - 2x + 1) = x(x-1)^2$; $P(x) = 0 \iff (x = 0 \text{ or } x = 1)$
Test Points: $P(-1) = (-)(+) < 0$; $P(\frac{1}{2}) = (+)(+) > 0$; $P(2) = (+)(+) > 0$



9. Division by $x - 1$ yields: $P(x) = (x-1)(x^2 + 5x + 4) = (x-1)(x+4)(x+1)$
Then: $P(x) = 0 \iff (x = 1 \text{ or } x = -4 \text{ or } x = -1)$
Test Points: $P(-5) = (-)(-)(-) < 0$; $P(-2) = (-)(+)(-) > 0$; $P(0) = (-)(+)(+) < 0$; $P(2) = (+)(+)(+) > 0$



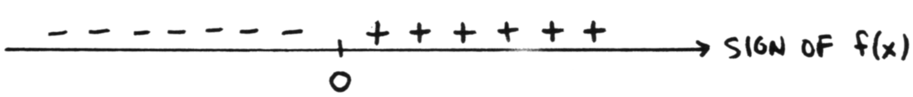
10. Division by $x + 1$ yields: $P(x) = (x+1)(x^2 - x - 12) = (x+1)(x-4)(x+3)$
Then: $P(x) = 0 \iff (x = -1 \text{ or } x = 4 \text{ or } x = -3)$
Test Points: $P(-4) = (-)(-)(-) < 0$; $P(-2) = (-)(-)(+) > 0$; $P(0) = (+)(-)(+) < 0$; $P(5) = (+)(+)(+) > 0$



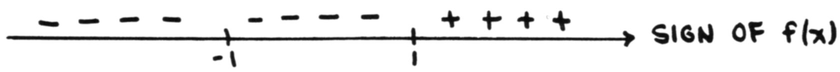
11. $f(x) = x^2(e^x - 1)$

$$\begin{aligned} f(x) = 0 &\iff x^2 = 0 \text{ or } e^x - 1 = 0 \\ &\iff x = 0 \text{ or } e^x = 1 \\ &\iff x = 0 \text{ or } x = 0 \\ &\iff x = 0 \end{aligned}$$

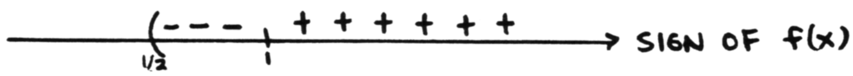
Test Points: $f(-1) = (-1)^2(e^{-1} - 1) = (+)(-) < 0$; $f(1) = 1^2(e^1 - 1) = (+)(+) > 0$



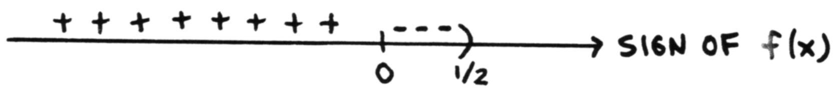
12. $f(x) = e^x(x^2 - 1) = e^x(x-1)(x+1)$; $\quad f(x) = 0 \iff (x = 1 \text{ or } x = -1)$
(Remember that e^x is never equal to 0.)
Test Points: $f(-2) = (+)(-)(+) < 0$; $f(0) = (+)(-)(+) < 0$; $f(2) = (+)(+)(+) > 0$



13. The only time a logarithm equals zero is when its input is 1.
Thus: $\ln(2x-1) = 0 \iff (2x - 1 = 1) \iff (x = 1)$. There are two test intervals: $(\frac{1}{2}, 1)$ and $(1, \infty)$
Test Points: $f(\frac{3}{4}) = \ln(2 \cdot \frac{3}{4} - 1) = \ln(\frac{1}{2}) < 0$; $f(2) = \ln(4-1) = \ln(3) > 0$



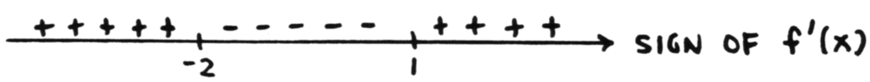
14. The only time a logarithm equals zero is when its input is 1.
Thus: $\ln(1-2x) = 0 \iff (1 - 2x = 1) \iff (x = 0)$. There are two test intervals: $(-\infty, 0)$ and $(0, \frac{1}{2})$
Test Points: $f(-1) = \ln(1+2) > 0$; $f(\frac{1}{4}) = \ln(1 - \frac{1}{4}) < 0$



15. Apply the 'number line test' to f'.

$$\begin{aligned} f'(x) &= 6x^2 + 6x - 12 \\ &= 6(x^2 + x - 2) \\ &= 6(x+2)(x-1) \\ f'(x) = 0 &\iff x = -2 \text{ or } x = 1 \end{aligned}$$

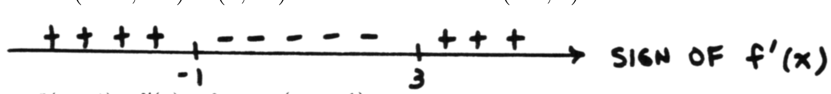
Test Points: $f'(-3) = (-)(-) > 0$; $f'(0) = (+)(-) < 0$; $f'(2) = (+)(+) > 0$
Thus: f increases on $(-\infty, -2) \cup (1, \infty)$ and decreases on $(-2, 1)$



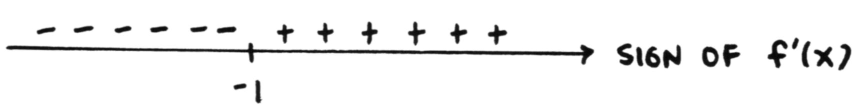
16. Apply the 'number line test' to f'.

$$\begin{aligned} f'(x) &= 3x^2 - 6x - 9 \\ &= 3(x^2 - 2x - 3 \\ &= 3(x-3)(x+1) \\ f'(x) = 0 &\iff x = 3 \text{ or } x = -1 \end{aligned}$$

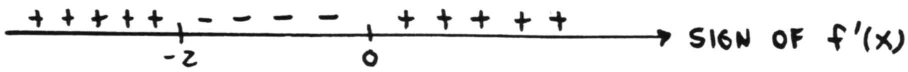
Test Points: $f'(-2) = (-)(-) > 0$; $f'(0) = (-)(+) < 0$; $f'(4) = (+)(+) > 0$
Thus: f increases on $(-\infty, -1) \cup (3, \infty)$ and decreases on $(-1, 3)$



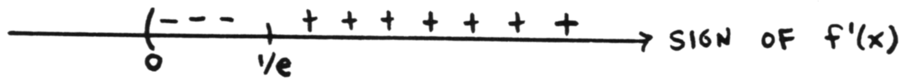
17. $f'(x) = xe^x + e^x = e^x(x+1)$; $\quad f'(x) = 0 \iff (x = -1)$
Test Points: $f'(-2) = (+)(-) < 0$; $f'(0) = (+)(+) > 0$
Thus: f decreases on $(-\infty, -1)$ and increases on $(-1, \infty)$



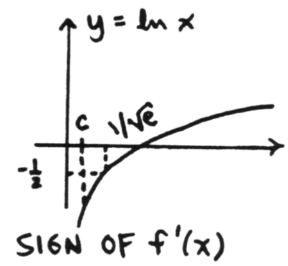
18. $f'(x) = x^2e^x + 2xe^x = xe^x(x+2)$; $f'(x) = 0 \iff (x = 0 \text{ or } x = -2)$
Test Points: $f'(-3) = (-)(+)(-) > 0$; $f'(-1) = (-)(+)(+) < 0$; $f'(1) = (+)(+)(+) > 0$
Thus: f increases on $(-\infty, -2) \cup (0, \infty)$ and decreases on $(-2, 0)$

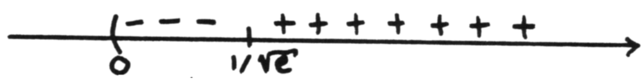


19. $f'(x) = x \cdot \frac{1}{x} + \ln x = 1 + \ln x$; $f'(x) = 0 \iff (\ln x = -1) \iff (x = e^{-1})$
Note that $\mathcal{D}(f) = (0, \infty)$. There are two test intervals: $(0, \frac{1}{e})$ and $(\frac{1}{e}, \infty)$
Test Points: $f'(\frac{1}{3}) = 1 + \ln(\frac{1}{3}) < 0$; $f'(1) = 1 + \ln 1 > 0$
Thus: f decreases on $(0, \frac{1}{e})$ and increases on $(\frac{1}{e}, \infty)$



20. $f'(x) = x^2 \cdot \frac{1}{x} + 2x \ln x = x(1 + 2\ln x)$
$f'(x) = 0 \iff (x = 0 \text{ or } 2\ln x = -1) \iff (x = 0 \text{ or } x = e^{-\frac{1}{2}})$
Here, $\mathcal{D}(f) = (0, \infty)$. Note that $x = 0$ is not in the domain of f.
Test Points: Choosing any number in $c \in (0, \frac{1}{\sqrt{e}})$, $\ln c < -\frac{1}{2}$, so that: $1 + 2\ln c < 0$
$f'(1) = (1)(1 + 2\ln 1) > 0$
Thus: f decreases on $(0, \frac{1}{\sqrt{e}})$ and increases on $(\frac{1}{\sqrt{e}}, \infty)$

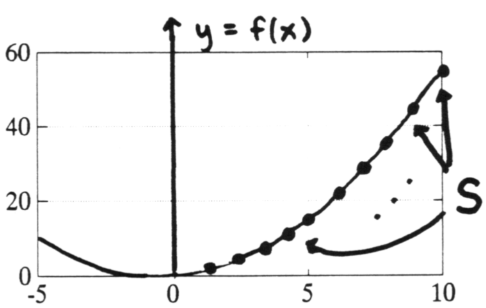




21. a) The solid dots are providing a picture of each summand. The first row has 1 dot, the second row 2 dots, the n^{th} row has n dots. The resulting 'triangular' shape is not easy to count; instead, duplicate this triangle (with the x's) to get a rectangular shape. The number of rows is n; the number of columns is $n+1$. Thus, the total number of solids dots and x's is $(n)(n+1)$; the number of solid dots is half this amount!
b) $1 + 2 + \cdots + 67 = \frac{(67)(67+1)}{2} = 2278$
c)

$$\begin{aligned} 64 + 65 + \cdots + 108 &= \overbrace{(1 + 2 + \cdots + 63) - (1 + 2 + \cdots + 63)}^{\text{add zero}} + (64 + 65 + \cdots + 108) \\ &= (1 + 2 + \cdots + 108) - (1 + 2 + \cdots + 63) \\ &= \frac{(108)(109)}{2} - \frac{(63)(64)}{2} \\ &= 5886 - 2016 = 3870 \end{aligned}$$

22. a) The function S is the same as the function f, except restricted to the positive integers. See the graph below.



b) $f'(x) = \frac{1}{2}[x(1) + (1)(x+1)] = \frac{1}{2}[2x+1] = x + \frac{1}{2}$
Note that:

$$\begin{aligned} f'(x) > 0 &\iff x + \frac{1}{2} > 0 \\ &\iff x > -\frac{1}{2} \end{aligned}$$

Thus: f increases on $(-\frac{1}{2}, \infty)$

23. a) $x \cdot S = x(1 + x + \cdots + x^n) = x + x^2 + \cdots + x^{n+1}$
b)

$$\begin{aligned} xS - S = &\qquad\quad x + x^2 + \cdots + x^n + x^{n+1} \\ &- (1 + x + x^2 + \cdots + x^n) \\ = &\; x^{n+1} - 1 \end{aligned}$$

Thus, $S \cdot (x-1) = x^{n+1} - 1$, so that solving for S yields:

$$S = \frac{x^{n+1} - 1}{x - 1}$$

c) Apply the formula with $x = 2$ and $n = 4$. Thus:

$$1 + 2 + 2^2 + 2^3 + 2^4 = \frac{2^{4+1} - 1}{2 - 1} = \frac{32 - 1}{1} = 31$$

Also: $1 + 2 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31$
d) Again, add zero in an appropriate form:

$$\begin{aligned} 2^6 + \cdots + 2^{10} &= (1 + 2 + 2^2 + \cdots + 2^{10}) - (1 + 2 + \cdots + 2^5) \\ &= \frac{2^{10+1} - 1}{2 - 1} - \frac{2^{5+1} - 1}{2 - 1} \\ &= (2^{11} - 1) - (2^6 - 1) \\ &= 2047 - 63 = 1984 \end{aligned}$$

Direct addition yields:

$$2^6 + 2^7 + 2^8 + 2^9 + 2^{10} = 64 + 128 + 256 + 512 + 1024 = 1984$$

24. a) If $0 < x < y$, then $0 < x^n < y^n$ for every positive integer n.
b) Letting $n = 2$, $S(x) = 1 + x + x^2$. Then, $S'(x) = 1 + 2x$; observe that $S'(x) = 0 \iff (x = -\frac{1}{2})$. A quick sketch of this line shows that $S'(x) > 0$ on $(-\frac{1}{2}, \infty)$, and $S'(x) < 0$ on $(-\infty, -\frac{1}{2})$. Thus, S increases on $(-\frac{1}{2}, \infty)$ and decreases on $(-\infty, -\frac{1}{2})$.
Of interest to us is that S increases on $(0, \infty)$. This shows that as x increases (for positive numbers x), $S(x)$ also increases, as suspected.
c) Letting $n = 3$, $S(x) = 1 + x + x^2 + x^3$. Then, $S'(x) = 1 + 2x + 3x^2$, which is always positive (check that the discriminant is negative). Thus, S increases everywhere.

25. a) There is an equal probability of getting each of the six sides. Letting $n = 1$: $P(1) = \frac{1}{6}(1) = \frac{1}{6}$
b) $P(2) = \frac{1}{6}[1 + (\frac{5}{6})^{2-1}] = \frac{1}{6}[1 + \frac{5}{6}] = \frac{1}{6} + (\frac{1}{6})(\frac{5}{6})$

c) The more times you throw the dice, the better the chance of getting a 2. Computing the first few values:

$$\begin{aligned} P(1) &= \frac{1}{6} \approx 0.1667 \\ P(2) &= \frac{1}{6}[1 + \frac{5}{6}] \approx 0.3056 \\ P(3) &= \frac{1}{6}[1 + \frac{5}{6} + (\frac{5}{6})^2] \approx 0.4213 \end{aligned}$$

d) Let $n \geq 1$. Since the number $(\frac{5}{6})^n$ is always positive:

$$\begin{aligned} P(n) &= \frac{1}{6}[1 + \frac{5}{6} + (\frac{5}{6})^2 + \cdots + (\frac{5}{6})^{n-1}] \\ &< \frac{1}{6}[1 + \frac{5}{6} + (\frac{5}{6})^2 + \cdots + (\frac{5}{6})^{n-1} + (\frac{5}{6})^n] \\ &:= P(n+1) \end{aligned}$$

Thus, $P(n) < P(n+1)$, so P is an increasing function.