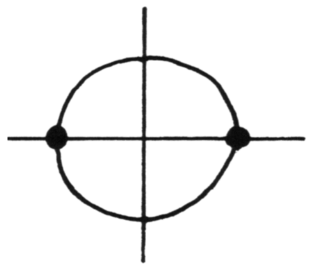
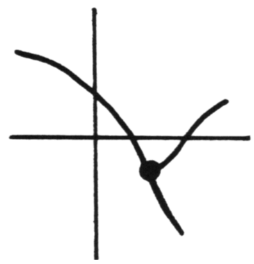
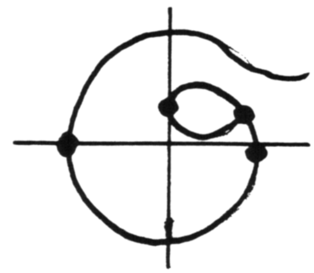
SECTION 4.8 Implicit Differentiation (Optional)

IN-SECTION EXERCISES:

EXERCISE 1.

The points where y is NOT locally a function of x are identified in the graphs below.

EXERCISE 2.

The final results are given in both *prime* and *Leibniz* notation.

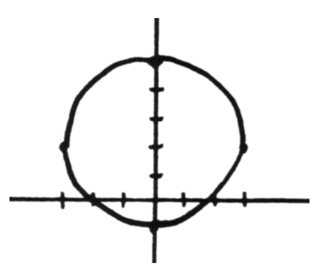
1. $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx} = 2yy'$
2. $\frac{d}{dx}(xy) = x\frac{dy}{dx} + (1)y = xy' + y$
3. $\frac{d}{dx}(x+y)^3 = 3(x+y)^2(1+\frac{dy}{dx}) = 3(x+y)^2(1+y')$
4. $\frac{d}{dx}(\ln y) = \frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{y}\cdot y'$

EXERCISE 3.

$$\begin{aligned}
y &= -(1-x^2)^{1/2} \\
\frac{dy}{dx} &= -\frac{1}{2}(1-x^2)^{-1/2}(-2x) \\
&= \frac{x}{\sqrt{1-x^2}} \\
&= -\frac{x}{-\sqrt{1-x^2}} \\
&= -\frac{x}{y}
\end{aligned}$$

EXERCISE 4.

1. The graph of $x^2 + (y-2)^2 = 3^2$ is the circle centered at $(0,2)$ with radius 3.



2. y is NOT locally a function of x at the points where there are vertical tangent lines: $(3,2)$ and $(-3,2)$

3. Differentiating implicitly, $2(y-2)^1\frac{dy}{dx}+2x=0$. Solving for $\frac{dy}{dx}$:

$$\begin{aligned}2(y-2)^1\frac{dy}{dx}+2x=0 \quad&\iff\quad 2(y-2)\frac{dy}{dx}=-2x\\ &\iff\quad \frac{dy}{dx}=\frac{-2x}{2(y-2)}\\ &\iff\quad \frac{dy}{dx}=-\frac{x}{y-2}=\frac{x}{2-y}\end{aligned}$$

Observe that the formula fails when $y=2$; these are precisely the points where y is NOT locally a function of x.

EXERCISE 5.

1. First, find the natural logarithm of y:

$$\begin{aligned}\ln y&=\ln\bigl(x^{-1}(2x-1)^{-1}(3x-1)^{-1}\bigr)\\ &=\ln x^{-1}+\ln(2x-1)^{-1}+\ln(3x-1)^{-1}\\ &=-\ln x-\ln(2x-1)-\ln(3x-1)\\ &=-(\ln x+\ln(2x-1)+\ln(3x-1))\end{aligned}$$

Then, differentiate implicitly:

$$\begin{aligned}\frac{1}{y}\cdot\frac{dy}{dx}&=(-1)\left(\frac{1}{x}+\frac{1}{2x-1}(2)+\frac{1}{3x-1}(3)\right)\\ &=-\left(\frac{1}{x}+\frac{2}{2x-1}+\frac{3}{3x-1}\right)\end{aligned}$$

Solving for $\frac{dy}{dx}$ yields:

$$\begin{aligned}\frac{dy}{dx}&=-y\left(\frac{1}{x}+\frac{2}{2x-1}+\frac{3}{3x-1}\right)\\ &=-\frac{1}{x(2x-1)(3x-1)}\left(\frac{1}{x}+\frac{2}{2x-1}+\frac{3}{3x-1}\right)\end{aligned}$$

2. You should be able to take the natural logarithm of y and simplify, all in one step:

$$\ln y=4\ln x+\frac{1}{3}\ln(x-1)-\frac{1}{5}\ln(2x+1)$$

Then, implicit differentiation yields

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{4}{x}+\frac{1}{3(x-1)}-\frac{2}{5(2x+1)}$$

so that:

$$\frac{dy}{dx}=\frac{x^4\sqrt[3]{x-1}}{\sqrt[5]{2x+1}}\cdot\left[\frac{4}{x}+\frac{1}{3(x-1)}-\frac{2}{5(2x+1)}\right]$$

EXERCISE 6.

1.

$$\begin{aligned}
\ln y &= x \ln x \\
\frac{1}{y} \cdot \frac{dy}{dx} &= x \cdot \frac{1}{x} + (1) \ln x \\
\frac{dy}{dx} &= x^x (1 + \ln x)
\end{aligned}$$

2.

$$\begin{aligned}
\ln y &= x \ln(2x) \\
\frac{1}{y} \cdot \frac{dy}{dx} &= x \cdot \frac{1}{2x} \cdot 2 + (1) \ln(2x) \\
\frac{dy}{dx} &= (2x)^x [1 + \ln(2x)]
\end{aligned}$$

3.

$$\begin{aligned}
\ln y &= 3x \ln(2x) \\
\frac{1}{y} \cdot \frac{dy}{dx} &= 3x \cdot \frac{1}{2x} \cdot 2 + (3) \ln(2x) \\
\frac{dy}{dx} &= (2x)^{3x} [3 + 3 \ln(2x)]
\end{aligned}$$

4.

$$\begin{aligned}
y &= [(x+1)^{1/2}]^{(x^2)} \\
&= (x+1)^{(\frac{1}{2}x^2)} \\
\ln y &= \frac{1}{2} x^2 \ln(x+1) \\
\frac{1}{y} \cdot \frac{dy}{dx} &= \frac{1}{2} x^2 \frac{1}{x+1} + (x) \ln(x+1) \\
\frac{dy}{dx} &= (\sqrt{x+1})^{(x^2)} [\frac{x^2}{2(x+1)} + x \ln(x+1)]
\end{aligned}$$

EXERCISE 7.

1. Let $a = 2$ and $b = -2$. Then the sentence '$a = b$' becomes '$2 = -2$', which is false. But the sentence '$a^2 = b^2$' becomes '$2^2 = (-2)^2$', which is true.
2. To show that a sentence of the form '$A \iff B$' is true, one can show that both '$A \implies B$' and '$B \implies A$' are true. This is justified by the logical equivalence:

$$A \iff B \quad \iff \quad [(A \implies B) \text{ and } (B \implies A)]$$

See the truth table below.

A	B	A ⇔ B	A ⇒ B	B ⇒ A	A ⇒ B AND B ⇒ A
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Proof. Let $a \geq 0$ and $b \geq 0$.

'$\Longrightarrow$'

Suppose that $a = b$. Then, $a^2 = b^2$. (This direction is true for *all* real numbers a and b.)

'$\Longleftarrow$'

Suppose that $a^2 = b^2$. Then, taking square roots (correctly!) yields $|a| = |b|$. Since a and b are nonnegative, $|a| = a$ and $|b| = b$. Thus, $a = b$.

3. Thus, whenever a and b are both known to be nonnegative, the sentence '$a = b$' always has the same truth value as the sentence '$a^2 = b^2$', so these two sentences can be used interchangeably.

END-OF-SECTION EXERCISES:

1. Completing the square:

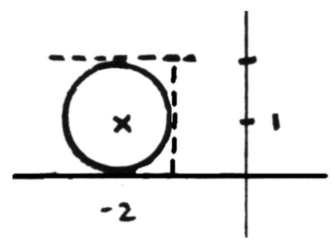
$$\begin{aligned} x^2 + 4x + y^2 - 2y + 4 = 0 \quad &\iff \quad (x^2 + 4x + (\frac{4}{2})^2) + (y^2 - 2y + (\frac{-2}{2})^2) = -4 + 4 + 1 \\ &\iff \quad (x+2)^2 + (y-1)^2 = 1 \end{aligned}$$

Rhe graph is the circle centered at $(-2, 1)$ with radius 1.

y is NOT locally a function of x at the points $(-1, 1)$ and $(-3, 1)$. (There are vertical tangent lines here.)

Differentiating implicitly:

$$2(x+2) + 2(y-1)y' = 0$$



Solving for y' yields:

$$y' = \frac{-2(x+2)}{2(y-1)} = \frac{-(x+2)}{y-1}$$

Note that the formula fails when $y = 1$.

The point $(-2, 2)$ lies on the circle since:

$$(-2+2)^2 + (2-1)^2 = 0^2 + 1^2 = 1$$

At this point, the tangent line has slope

$$y'|_{(-2,2)} = \frac{-(-2+2)}{2-1} = 0 \ ,$$

so there is a horizontal tangent line (as expected). The equation of the tangent line at the point $(-2, 2)$ is $y = 2$.

The point $(-1, 1)$ lies on the circle since:

$$(-1+2)^2 + (1-1)^2 = 1^2 + 0^2 = 1$$

At this point, the formula for y' fails; the tangent line is vertical, and has equation $x = -1$.

2. Same circle as (1), hence the same derivative. At $(-2, 0)$ there is a horizontal tangent line, with equation $y = 0$. At $(-3, 1)$ there is a vertical tangent line, with equation $x = -3$.

3. Put the circle in standard form, by completing the square:

$$\begin{aligned} 4x - 2y = -x^2 - y^2 - 1 &\iff x^2 + 4x + y^2 - 2y = -1 \\ &\iff (x^2 + 4x + 4) + (y^2 - 2y + 1) = -1 + 4 + 1 \\ &\iff (x+2)^2 + (y-1)^2 = 2^2 \end{aligned}$$

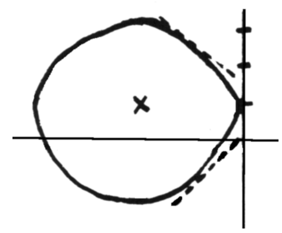
The graph is the circle centered at $(-2, 1)$ with radius 2.
y is NOT locally a function of x at the points $(0, 1)$ and $(-4, 1)$; there are vertical tangent lines here.
Differentiating implicitly:

$$2(x+2) + 2(y-1)y' = 0$$

Solving for y' yields:

$$y' = -\frac{2(x+2)}{2(y-1)} = -\frac{x+2}{y-1}$$

Note that the formula fails when $y = 1$.



The point $(-1, 1+\sqrt{3})$ lies on the circle since:

$$(-1+2)^2 + (1+\sqrt{3}-1)^2 = 1 + (\sqrt{3})^2 = 1 + 3 = 4$$

The slope of the tangent line at this point is:

$$y'|_{(-1,1+\sqrt{3})} = -\frac{-1+2}{1+\sqrt{3}-1} = -\frac{1}{\sqrt{3}}$$

The equation of the tangent line is:

$$y - (1+\sqrt{3}) = -\frac{1}{\sqrt{3}}(x - (-1))$$

4. Same circle as (3); hence the same derivative. The point $(-1, 1-\sqrt{3})$ lies on the graph since:

$$(-1+2)^2 + (1-\sqrt{3}-1)^2 = 1 + (-\sqrt{3})^2 = 1 + 3 = 4$$

The slope of the tangent line at this point is:

$$y'|_{(-1,1-\sqrt{3})} = -\frac{-1+2}{1-\sqrt{3}-1} = -\frac{1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$$

The equation of the tangent line is:

$$y - (1-\sqrt{3}) = \frac{1}{\sqrt{3}}(x - (-1))$$