SECTION 4.6 Differentiating Products and Quotients

IN-SECTION EXERCISES:

EXERCISE 1.

Most any functions f and g that you might 'pull out the air' will serve to show that, in general, the derivative of a quotient is NOT the quotient of the derivatives. For example, choose $f(x) = 2$ and $g(x) = x$. Then, $(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{2}{x} = 2x^{-1}$, so that:

$$(\frac{f}{g})'(x) = \frac{d}{dx}2x^{-1} = -2x^{-2} = -\frac{2}{x^2}$$

However:

$$\frac{f'(x)}{g'(x)} = \frac{0}{1} = 0$$

Note that:

$$(\frac{f}{g})'(x) \neq \frac{f'(x)}{g'(x)}$$

EXERCISE 2.

Rewrite the proof several times, without looking at the book. Be sure that you can justify each step in your proof. In particular, make sure that you can explain why $\lim_{h\to 0} f(x+h)$ is equal to $f(x)$.

EXERCISE 3.

First, multiply f out and differentiate term-by-term:

$$\begin{aligned} f(x) &= 2x^4 + 8x^3 - x - 4 \\ f'(x) &= 8x^3 + 24x^2 - 1 \end{aligned}$$

Next, use the product rule:

$$\begin{aligned} f'(x) &= (x+4)(6x^2) + (1)(2x^3 - 1) \\ &= 6x^3 + 24x^2 + 2x^3 - 1 \\ &= 8x^3 + 24x^2 - 1 \end{aligned}$$

Compare!

EXERCISE 4.

1. $(abcde)' = a'bcde + ab'cde + abc'de + abcd'e + abcde'$
2. $f'(x) = (2)(x^2 - 3)(4 - x) + (2x + 1)(2x)(4 - x) + (2x + 1)(x^2 - 3)(-1)$

EXERCISE 5.

1. Using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(2x-1)(1)-(x)(2)}{(2x-1)^2} \\ &= \frac{-1}{(2x-1)^2} \end{aligned}$$

To differentiate f without using the quotient rule, it must first be rewritten:

$$\begin{aligned} f(x) &= x(2x-1)^{-1} \\ f'(x) &= x[-(2x-1)^{-2}(2)] + (1)(2x-1)^{-1} \\ &= \frac{-2x}{(2x-1)^2} + \frac{1}{2x-1} \\ &= \frac{-2x}{(2x-1)^2} + \frac{1}{2x-1}\cdot\frac{(2x-1)}{(2x-1)} \\ &= \frac{-2x+(2x-1)}{(2x-1)^2} \\ &= \frac{-1}{(2x-1)^2} \end{aligned}$$

Which way was easier?

2. Using the quotient rule:

$$\begin{aligned} g'(x) &= \frac{(1-x)^4(0) - 3\cdot 4(1-x)^3(-1)}{\big((1-x)^4\big)^2} \\ &= \frac{12(1-x)^3}{(1-x)^8} \\ &= \frac{12}{(1-x)^5} \end{aligned}$$

To differentiate g without using the quotient rule, it must first be rewritten:

$$\begin{aligned} g(x) &= 3(1-x)^{-4} \\ g'(x) &= 3[-4(1-x)^{-5}(-1) + (0)(1-x)^{-4} \\ &= \frac{12}{(1-x)^5} \end{aligned}$$

This time, which way was easier?

EXERCISE 6.

1. When $n = 1$, the formula holds:

$$1 = \frac{1(1+1)}{2}$$

Suppose that the formula holds when $n = K$; that is, suppose it is true that:

$$1 + 2 + \cdots + K = \frac{K(K+1)}{2}$$

Now, it must be shown that the formula is true when $n = K + 1$:

$$\begin{aligned}
1 + 2 + \cdots + K + (K+1) &= [1 + 2 + \cdots + K] + (K+1) && \text{(regroup)}\\
&= \frac{K(K+1)}{2} + (K+1) && \text{(inductive hypothesis)}\\
&= \frac{K(K+1)}{2} + \frac{2(K+1)}{2} && \text{(get common denominator)}\\
&= \frac{K(K+1) + 2(K+1)}{2} && \text{(add fractions)}\\
&= \frac{(K+1)(K+2)}{2} && \text{(factor out } K+1\text{)}
\end{aligned}$$

Thus, the formula holds when n is replaced by $K + 1$. Therefore, the formula holds for all positive integers.

2. Taking $n = 512$:

$$1 + 2 + \cdots + 512 = \frac{(512)(512+1)}{2} = \frac{(512)(513)}{2} = 131,328$$

3. The trick is to add zero in an appropriate form:

$$\begin{aligned}
100 + 101 + \cdots + 512 &= (1 + 2 + \cdots + 99) + (100 + \cdots + 512) - (1 + 2 + \cdots + 99)\\
&= (1 + 2 + \cdots + 512) - (1 + 2 + \cdots + 99)\\
&= \frac{(512)(513)}{2} - \frac{(99)(100)}{2}\\
&= 131,328 - 4,950 = 126,378
\end{aligned}$$

END-OF-SECTION EXERCISES:

1.

$$\begin{aligned}
y' &= x \cdot 3(2-x)^2(-1) + (1)(2-x)^3\\
&= -3x(2-x)^2 + (2-x)^3\\
&= (2-x)^2[-3x + (2-x)]\\
&= (2-x)^2[2-4x]\\
&= 2(2-x)^2(1-2x)
\end{aligned}$$

$$\begin{gathered}
y(0) = 0 \cdot (2-0)^3 = 0\\
y(t^2) = t^2(2-t^2)^3\\
y'(0) = 2(2-0)^2(1-2(0)) = 2(4)(1) = 8\\
y'(t) = 2(2-t)^2(1-2t)
\end{gathered}$$

2.

$$\begin{aligned}
y &= e^{-x}(x^2-1)^{\frac{1}{2}}\\
y' &= e^{-x}\frac{1}{2}(x^2-1)^{-\frac{1}{2}}(2x) + e^{-x}(-1)(x^2-1)^{\frac{1}{2}}\\
&= \frac{xe^{-x}}{\sqrt{x^2-1}} - e^{-x}\sqrt{x^2-1}\cdot\frac{\sqrt{x^2-1}}{\sqrt{x^2-1}}\\
&= \frac{xe^{-x} - e^{-x}(x^2-1)}{\sqrt{x^2-1}}\\
&= \frac{e^{-x}(x-(x^2-1))}{\sqrt{x^2-1}}\\
&= \frac{e^{-x}(-x^2+x+1)}{\sqrt{x^2-1}}
\end{aligned}$$

$$y(1) = e^{-1}\sqrt{1^2-1} = 0$$

$$y'(1) \text{ does not exist}$$

$$\frac{dy}{dx}\Big|_{x=\sqrt{2}} = y'(\sqrt{2}) = \frac{e^{-\sqrt{2}}(-2+\sqrt{2}+1)}{\sqrt{(\sqrt{2})^2-1}} = e^{-\sqrt{2}}(\sqrt{2}-1)$$

$$y(1)\cdot y'(\sqrt{2}) = (0)\cdot(y'(\sqrt{2})) = 0$$

3.

$$\begin{aligned}
f'(x) &= e^x\cdot\frac{1}{x} + e^x\ln x\\
&= e^x(\frac{1}{x}+\ln x)
\end{aligned}$$

$$\mathcal{D}(f) = (0,\infty)$$

$$\mathcal{D}(f') = (0,\infty)$$

$$f'(e^x) = e^{(e^x)}(\frac{1}{e^x}+\ln e^x) = e^{(e^x)}(\frac{1}{e^x}+x)$$

$$f'(e^2) = e^{(e^2)}(\frac{1}{e^2}+2)$$

4.

$$f'(x) = \frac{1}{\ln x}\cdot\frac{1}{x} = \frac{1}{x\ln x}$$

$$\mathcal{D}(f) = \{x \mid x>0 \text{ and } \ln x > 0\} = (1,\infty)$$

$$\mathcal{D}(f') = \{x \mid x\in\mathcal{D}(f) \text{ and } x>0 \text{ and } \ln x\neq 0 \text{ and } x\neq 0\} = (1,\infty)$$

$$f'(e^x) = \frac{1}{e^x\ln e^x} = \frac{1}{xe^x}$$

$$f'(f(e)) = f'(\ln(\ln e)) = f'(\ln 1) = f'(0) \text{ does not exist, since } 0\notin\mathcal{D}(f')$$

5.

$$
\begin{gathered}
g'(x) = e^{(e^x)} \cdot e^x = e^{x+e^x} \\
\lim_{x\to 0} g(x) = \lim_{x\to 0} e^{(e^x)} = e^{(e^0)} = e^1 = e \\
\lim_{x\to 0} g'(x) = \lim_{x\to 0} e^{x+e^x} = e^{0+e^0} = e^1 = e \\
\mathcal{D}(g) = \mathbb{R} \\
g(g'(g(0))) = g(g'(e^1)) = g(e^{e+e^e}) = e^{e^{(e^{e+e^e})}}
\end{gathered}
$$

6.

$$
\begin{aligned}
g'(x) &= (1)(2x+1)(1-x)^7 + (x-1)(2)(1-x)^7 + (x-1)(2x+1)7(1-x)^6(-1) \\
&= (2x+1)(1-x)^7 + 2(x-1)(1-x)^7 - 7(x-1)(2x+1)(1-x)^6
\end{aligned}
$$

$$
\begin{gathered}
\lim_{x\to 0} g(x) = (0-1)(2(0)+1)(1-0)^7 = (-1)(1)(1) = -1 \\
\lim_{t\to 0} g(t) = -1 \\
\lim_{x\to 0} g'(x) = (1)(1) + 2(-1)(1) - 7(-1)(1)(1) = 1 - 2 + 7 = 6 \\
g'(0) = 6 \\
g' \text{ IS continuous at 0, since } \lim_{x\to 0} g'(x) = g'(0)
\end{gathered}
$$

7. There are several correct approaches. If desired, h can first be rewritten as $h(x) = \ln e^x - \ln(x+1) = x - \ln(x+1)$, and then differentiated. Or, h can be differentiated directly:

$$
\begin{aligned}
h'(x) &= \frac{x+1}{e^x}\left[\frac{(x+1)e^x - e^x(1)}{(x+1)^2}\right] \\
&= \frac{1}{e^x}\left[\frac{e^x(x+1-1)}{x+1}\right] \\
&= \frac{x}{x+1}
\end{aligned}
$$

When $x = 0$, $h(0) = \ln(\frac{e^0}{0+1}) = \ln(1) = 0$. The slope of the tangent line at the point $(0,0)$ is $h'(0) = \frac{0}{0+1} = 0$. Thus, the tangent line is horizontal, and has equation $y = 0$.

8.

$$
\begin{aligned}
h(x) &= \sqrt{\ln x^3} = (3\ln x)^{1/2} = \sqrt{3}(\ln x)^{1/2} \\
h'(x) &= \sqrt{3} \cdot \frac{1}{2}(\ln x)^{-1/2} \cdot \frac{1}{x} \\
&= \frac{\sqrt{3}}{2x\sqrt{\ln x}}
\end{aligned}
$$

$$
\mathcal{D}(h) = \{x \mid x > 0 \text{ and } \ln x \ge 0\} = \{x \mid x \ge 1\}
$$

When $x = e$, $h(e) = \sqrt{3}$, so the point $(e, \sqrt{3})$ lies on the graph of h. The slope of the tangent line at this point is:

$$
h'(e) = \frac{\sqrt{3}}{2e\sqrt{\ln e}} = \frac{\sqrt{3}}{2e}
$$

The tangent line has equation $y - \sqrt{3} = \frac{\sqrt{3}}{2e}(x - e)$.

9.

$$\begin{aligned} f'(x) &= e^{2x}7(2x+1)^6 \cdot 2 + 2e^{2x}(2x+1)^7 \\ &= 2e^{2x}(2x+1)^6[7+(2x+1)] \\ &= 2e^{2x}(2x+1)^6[8+2x] \\ &= 4e^{2x}(2x+1)^6(x+4) \end{aligned}$$

When $x = 0$, $f(0) = e^{2\cdot 0}(2\cdot 0+1)^7 = 1$, so the point $(0,1)$ lies on the graph of f. The slope of the tangent line at this point is:

$$f'(0) = 4e^{2\cdot 0}(2\cdot 0+1)^6(0+4) = 4(4) = 16$$

The tangent line has equation $y - 1 = 16(x-0)$, or, equivalently, $y = 16x + 1$.

10. Using a generalized product rule:

$$f'(x) = 2(ax+b)(a)(cx+d)^3(x+1)^4 + (ax+b)^2 3(cx+d)^2(c)(x+1)^4 + (ax+b)^2(cx+d)^3 4(x+1)^3(1)$$

11.

$$\begin{aligned} h(t) &= e(3t-1)^{-4} \\ h'(t) &= -4e(3t-1)^{-5}(3) \\ &= \frac{-12e}{(3t-1)^5} \end{aligned}$$

When $t = \frac{2}{3}$, $h(\frac{2}{3}) = e(3(\frac{2}{3})-1)^{-4} = e$, so the point $(\frac{2}{3}, e)$ lies on the graph of h. The slope of the tangent line at this point is:

$$h'(\frac{2}{3}) = \frac{-12e}{1} = -12e$$

So, the tangent line has equation $y - e = -12e(t - \frac{2}{3})$.

12. Using the Quotient Rule:

$$\begin{aligned} y' &= \frac{\sqrt{t+2}(\frac{1}{t}) - \ln t \cdot \frac{d}{dt}(\sqrt{t+2})}{t+2} \\ &= \frac{\frac{\sqrt{t+2}}{t} - \ln t \cdot \frac{1}{2}(t+2)^{-1/2}(1)}{t+2} \\ &= \frac{\frac{\sqrt{t+2}}{t} - \frac{\ln t}{2\sqrt{t+2}}}{t+2} \\ &= \frac{\frac{\sqrt{t+2}}{t} \cdot \frac{2\sqrt{t+2}}{2\sqrt{t+2}} - \frac{\ln t}{2\sqrt{t+2}} \cdot \frac{t}{t}}{t+2} \\ &= \frac{2(t+2) - t\ln t}{2t\sqrt{t+2}(t+2)} \\ &= \frac{2(t+2) - t\ln t}{2t\sqrt{(t+2)^3}} \end{aligned}$$

The instantaneous rate of change at $t = 1$ is:

$$y'(1) = \frac{2(1+2) - 1\ln 1}{2(1)\sqrt{(1+2)^3}} = \frac{2(3)}{2\sqrt{27}} = \frac{3}{\sqrt{9\cdot 3}} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$$

13.

$$\begin{aligned} y' &= 2[(x-3)(x+1)(2x-1)]^1[(1)(x+1)(2x-1)+(x-3)(1)(2x-1)+(x-3)(x+1)(2)] \\ &= \cdots \\ &= 8(x-3)(x+1)(2x-1)(x^2-3x-2) \end{aligned}$$

Using the quadratic formula:

$$\begin{aligned} x^2-3x-2=0 \quad &\iff \quad x = \frac{3 \pm \sqrt{9-4(1)(-2)}}{2(1)} \\ &\iff \quad x = \frac{3 \pm \sqrt{17}}{2} \end{aligned}$$

The tangent line is horizontal if and only if the first derivative equals zero. Therefore:

$$\begin{aligned} y'=0 \quad &\iff \quad x-3=0 \ \text{ or } \ x+1=0 \ \text{ or } \ 2x-1=0 \ \text{ or } \ x^2-3x-2=0 \\ &\iff \quad x=3 \text{ or } x=-1 \text{ or } x=\frac{1}{2} \text{ or } x=\frac{3\pm\sqrt{17}}{2} \end{aligned}$$