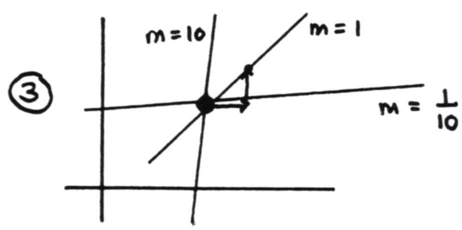
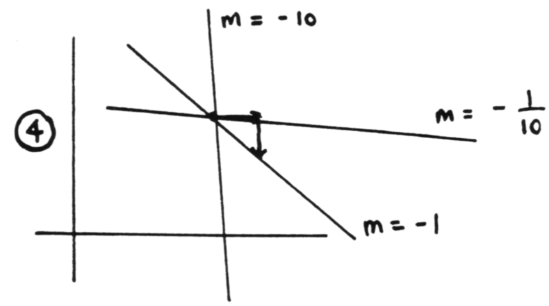
SECTION 4.1 Tangent Lines

IN-SECTION EXERCISES:

EXERCISE 1.

1. $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{(-1)(y_1 - y_2)}{(-1)(x_1 - x_2)} = \dfrac{y_1 - y_2}{x_1 - x_2}$
2. slope 3: when the x-values differ by 1, the y-values differ by 3. When the x-values differ by 2, the y-values differ by $(2)(3) = 6$.
3. The graph is given below:





4. The graph is given above:

EXERCISE 2.

1. Let $f(x) = -3x$. Then:

$$\begin{aligned}\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h\to 0} \frac{-3(1+h) - (-3)}{h} \\ &= \lim_{h\to 0} \frac{-3 - 3h + 3}{h} \\ &= \lim_{h\to 0} \frac{-3h}{h} = -3\end{aligned}$$

Thus, the slope of the tangent line at the point $(1, -3)$ is -3.

2. Let $f(x) = -3x$. Then:

$$\begin{aligned}\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{-3(x+h) - (-3x)}{h} \\ &= \lim_{h\to 0} \frac{-3x - 3h + 3x}{h} \\ &= \lim_{h\to 0} \frac{-3h}{h} = -3\end{aligned}$$

Thus, the slope of the tangent line at the point $(x, f(x))$ is -3.

3. Let $f(x) = kx$. Then:

$$\begin{aligned}\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{k(x+h) - kx}{h} \\ &= \lim_{h\to 0} \frac{kx + kh - kx}{h} \\ &= \lim_{h\to 0} \frac{kh}{h} = k\end{aligned}$$

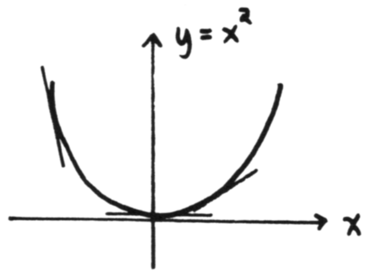
Thus, the slope of the tangent line at the point $(x, f(x))$ is k.

4. Let $f(x) = 0$. Then:

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{0 - 0}{h} = 0$$

EXERCISE 3.

1. The graph of $f(x) = x^2$ is shown below:



2. When $x = 0$, the slope of the tangent line is zero.

   When x is a small positive number, the slope is a small positive number.

   When x is a large negative number, the slope is a large negative number.

3. Let $f(x) = x^2$. Then:

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0}\frac{(x^2+2xh+h^2)-x^2}{h} \\ &= \lim_{h\to 0}\frac{h(2x+h)}{h} \\ &= \lim_{h\to 0}(2x+h) = 2x\end{aligned}$$

4. Thus, at a point (x, x^2), the slope of the tangent line exists, and is equal to twice the x-value of the point. This certainly agrees with our expectations. When $x = 0$, $2(0) = 0$. When x is a small positive number, so is $2x$. And when x is a large negative number, so is $2x$.

EXERCISE 4.

1. The 'left-hand' line contains points $(-2, -3)$ and $(1, 3)$; thus, it has slope $\frac{3-(-3)}{1-(-2)} = 2$. Using point-slope form (see Algebra Review, this section), the equation of the line is:

$$y - 3 = 2(x-1) \quad\iff\quad y = 3 + 2(x-1) \quad\iff\quad y = 2x+1$$

   The 'right-hand' line contains points $(4, 1.5)$ and $(1, 3)$; thus, it has slope $\frac{3-1.5}{1-4} = \frac{1.5}{-3} = -\frac{1}{2}$. Using point-slope form, the equation of the line is:

$$y - 3 = -\frac{1}{2}(x-1) \quad\iff\quad y = 3 - \frac{1}{2}(x-1) \quad\iff\quad y = -\frac{1}{2}x + \frac{7}{2}$$

   Thus:

$$f(x) = \begin{cases} 2x+1 & \text{for } x \le 1 \\ -\frac{1}{2}x + \frac{7}{2} & \text{for } x > 1 \end{cases}$$

2. We had better find that the right-hand limit equals $-\frac{1}{2}$. (♣ Why?) Indeed, when $h > 0$, then $1+h > 1$, so that:

$$\begin{aligned}\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0^+}\frac{(-\frac{1}{2}(1+h)+\frac{7}{2}-3)}{h} \\ &= \lim_{h\to 0^+}\frac{-\frac{1}{2}h}{h} = -\frac{1}{2}\end{aligned}$$

3. We had better find that the left-hand limit equals 2. (♣ Why?) Indeed, when $h < 0$, then $1 + h < 1$, so that:

$$\begin{aligned}\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0^-} \frac{(2(1+h)+1-3)}{h} \\ &= \lim_{h\to 0^-} \frac{2h}{h} = 2\end{aligned}$$

4. Since the one-sided limits do not agree, the two-sided limit does not exist. There is no tangent line at the point $(1, 3)$, as expected.

EXERCISE 5.

1. If f is defined on both sides of x, then $f(x + h)$ is defined both for $h > 0$ and for $h < 0$. The limit is a genuine two-sided limit in this case.
2. If f is only defined to the right of x, then $f(x+h)$ is only defined for $h > 0$. We can only let h approach 0 from the right-hand side. Thus, in this case, the 'two-sided' limit is actually a right-hand limit:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$$

3. If f is only defined to the left of x, then $f(x+h)$ is only defined for $h < 0$. We can only let h approach 0 from the left-hand side. Thus, in this case, the 'two-sided' limit is actually a left-hand limit:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$$

EXERCISE 6.

$$\begin{aligned}y-(-2) = -\frac{5}{6}(x-5) \quad &\iff \quad y+2 = -\frac{5}{6}x + \frac{25}{6} \\ &\iff \quad y = -\frac{5}{6}x + \frac{13}{6}\end{aligned}$$

Also:

$$\begin{aligned}y-3 = -\frac{5}{6}(x-(-1)) \quad &\iff \quad y-3 = -\frac{5}{6}x - \frac{5}{6} \\ &\iff \quad y = -\frac{5}{6}x + \frac{13}{6}\end{aligned}$$

Combining results:

$$y-(-2) = -\frac{5}{6}(x-5) \quad \iff \quad y = -\frac{5}{6}x + \frac{13}{6} \quad \iff \quad y-3 = -\frac{5}{6}(x-(-1))$$

Thus, the two equations are true at precisely the same times; they describe the same line.

END-OF-SECTION EXERCISES:

1. EXP; when the limit exists, it is a number that tells the slope of the tangent line to the graph of f at the point $(x, f(x))$.
2. EXP; when the limit exists, it is a number that tells the slope of the tangent line to the graph of g at the point $(x, g(x))$.
3. SEN; CONDITIONAL. The truth depends upon the choices made for the function f, the number $x \in \mathcal{D}(f)$, and the number m.
4. SEN; CONDITIONAL. The truth depends upon the choices made for the function g, the number $x \in \mathcal{D}(g)$, and the number m.

5. SEN; TRUE. See the in-section Exercise 3.

6. SEN; TRUE. The graph of g is a horizontal line; the slope of every tangent line is 0.

7. $g(0.1) = \dfrac{f(x+0.1) - f(x)}{0.1}$; $g(\Delta x) = \dfrac{g(x+\Delta x) - g(x)}{\Delta x}$

8. $\displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} g(h)$

9. There are two things to 'worry' about; h cannot be zero (since this would produce division by 0), and $x+h$ must be in the domain of f. Therefore:

$$h \in \mathcal{D}(g) \quad\iff\quad h \neq 0 \ \text{ and } \ x+h \in \mathcal{D}(f)$$

10. The number $g(h)$ tells the slope of the secant line through the points $(x, f(x))$ and $(x+h, f(x+h))$.

11. When $\lim_{h\to 0} g(h)$ exists, it tells the slope of the tangent line to the graph of f at the point $(x, f(x))$.

12. Remember that ' $\iff$ ' can also be written as 'if and only if'.
$\lim_{h\to 0} g(h) = m$ if and only if, for every $\epsilon > 0$, there exists $\delta > 0$, such that whenever $h \in \mathcal{D}(g)$ and $0 < |h - 0| < \delta$, then $|g(h) - m| < \epsilon$