SECTION 3.2 Limits—Making it Precise

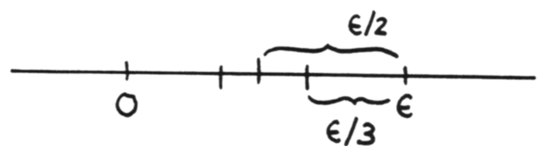
IN-SECTION EXERCISES:

EXERCISE 1.

1. Whenever x is within $\frac{\epsilon}{2}$ of 3, then $f(x)$ will be within ϵ of 6 . We can certainly restrict x to any smaller interval about 3, if desired. That is, δ can be taken to be *any* positive number less than $\frac{\epsilon}{2}$.
2. $\frac{\epsilon}{2} - .01$ is not necessarily positive; indeed:

$$\begin{aligned} \frac{\epsilon}{2} - .01 > 0 \quad &\iff \quad \frac{\epsilon}{2} > .01 \\ &\iff \quad \epsilon > .02 \end{aligned}$$

   In particular, if $\epsilon < .02$, then $\frac{\epsilon}{2} - .01$ is negative!
3. As long as ϵ is a positive number, $\frac{\epsilon}{3}$ is always less than $\frac{\epsilon}{2}$.



4. Every number $\frac{\epsilon}{n}$, where $n > 2$, is less than $\frac{\epsilon}{2}$.

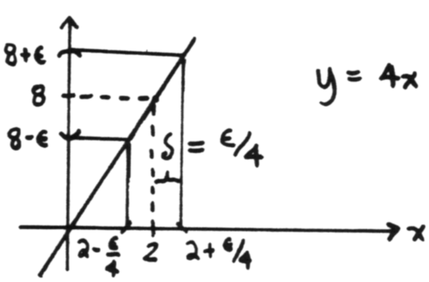
EXERCISE 2.

1. Step 1. It must be shown that we can get $4x$ as close to 8 as desired, by requiring that x be sufficiently close to 2 .

   Step 2. Let $\epsilon > 0$; we want to get $4x$ in the interval $(8 - \epsilon, 8 + \epsilon)$.

   Step 3. When $y = 8 - \epsilon$, we have $x = \frac{8-\epsilon}{4} = 2 - \frac{\epsilon}{4}$. When $y = 8 + \epsilon$, we have $x = \frac{8+\epsilon}{4} = 2 + \frac{\epsilon}{4}$. (Note that for this line, when x changes by a given amount, y changes by 4 times that amount. Equivalently, when y changes by some amount, x changes by one-fourth that amount.)

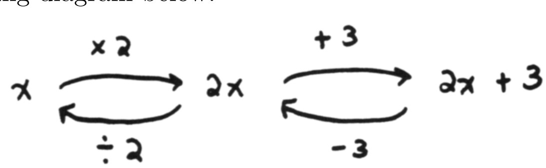
   Step 4. Take $\delta = \frac{\epsilon}{4}$. Then, whenever x is within δ of 2, $4x$ is within ϵ of 8 .



2. Step 1. It must be shown that we can get $2x + 3$ as close to 5 as desired, by requiring that x be sufficiently close to 1 .

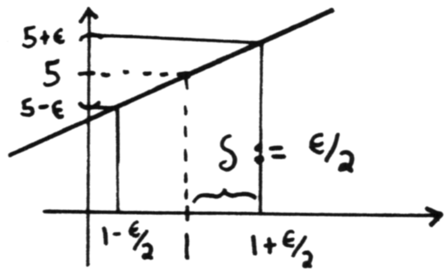
   Step 2. Let $\epsilon > 0$; we want to get $2x + 3$ in the interval $(5 - \epsilon, 5 + \epsilon)$.

   Step 3. Refer to the mapping diagram below.



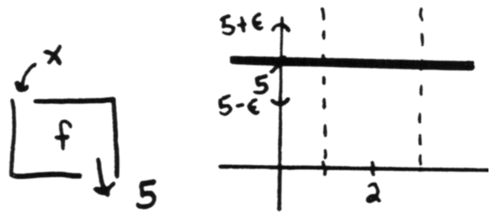
   When $y = 5 - \epsilon$, we have $x = \frac{(5-\epsilon)-3}{2} = 1 - \frac{\epsilon}{2}$. When $y = 5 + \epsilon$, we have $x = \frac{(5+\epsilon)-3}{2} = 1 + \frac{\epsilon}{2}$. (Note that for this line, when x changes by a given amount, y changes by 2 times that amount. Equivalently, when y changes by some amount, x changes by one-half that amount. Thus, when y changes by ϵ, x changes by $\frac{\epsilon}{2}$.)

Step 4. Take $\delta = \frac{\epsilon}{2}$. Then, whenever x is within δ of 1, $2x + 3$ is within ϵ of 5.



EXERCISE 3.

1. The function f takes an input x, and outputs the number 5.
2. The 'black box' below describes f.



3. $\lim_{x\to 2} f(x) = 5$; when x is close to 2, $f(x)$ is close to 5
4. The 4-step process can be greatly simplified for this function. We need to get the function values within ϵ of 5, by requiring that x be close enough to 2. But the function values are *always* 5, regardless of what x is! So, we can take δ to be *any* positive number. For example, take $\delta := 1$. Then, when x is within 1 unit of 2, $f(x) = 5$ is within ϵ of 5!
5. We can take δ to be *any* positive number!

EXERCISE 4.

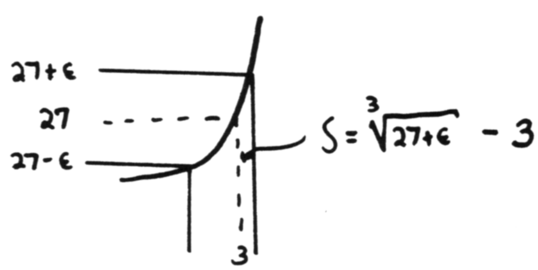
1. Step 1. It must be shown that we can get x^3 as close to 27 as desired, by requiring that x be sufficiently close to 3.

   Step 2. Let $\epsilon > 0$. We want to get x^3 in the interval $(27 - \epsilon, 27 + \epsilon)$.

   Step 3. Refer to the mapping diagram at right.

   When $y = 27 - \epsilon$, we have $x = \sqrt[3]{27 - \epsilon}$. When $y = 27 + \epsilon$, we have $x = \sqrt[3]{27 + \epsilon}$.

   Step 4. Take $\delta := \sqrt[3]{27 + \epsilon} - 3$, since this is the shorter distance. Then, whenever x is within δ of 3, x^3 is within ϵ of 27.

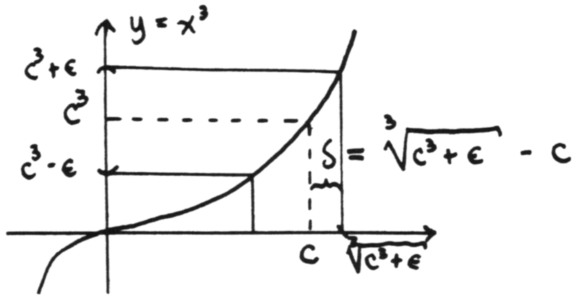


2. Step 1. It must be shown that we can get x^3 as close to c^3 as desired, by requiring that x be sufficiently close to c.

   Step 2. Let $\epsilon > 0$. We want to get x^3 in the interval $(c^3 - \epsilon, c^3 + \epsilon)$.

   Step 3. When $y = c^3 - \epsilon$, we have $x = \sqrt[3]{c^3 - \epsilon}$. When $y = c^3 + \epsilon$, we have $x = \sqrt[3]{c^3 + \epsilon}$.

Step 4. Take $\delta := \sqrt[3]{c^3+\epsilon} - c$, since this is the shorter distance. Then, whenever x is within δ of c, x^3 is within ϵ of c^3.



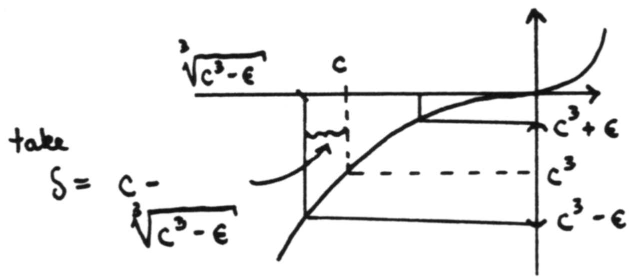
EXERCISE 5.

1. Step 1. Let $c < 0$. It must be shown that we can get x^3 as close to c^3 as desired, by requiring that x be sufficiently close to c.

   Step 2. Let $\epsilon > 0$. We want to get x^3 in the interval $(c^3 - \epsilon, c^3 + \epsilon)$.

   Step 3. When $y = c^3 - \epsilon$, we have $x = \sqrt[3]{c^3-\epsilon}$. When $y = c^3 + \epsilon$, we have $x = \sqrt[3]{c^3+\epsilon}$.

   Step 4. Take $\delta := c - \sqrt[3]{c^3-\epsilon}$, since now THIS is the shorter distance. Then, whenever x is within δ of c, x^3 is within ϵ of c^3.

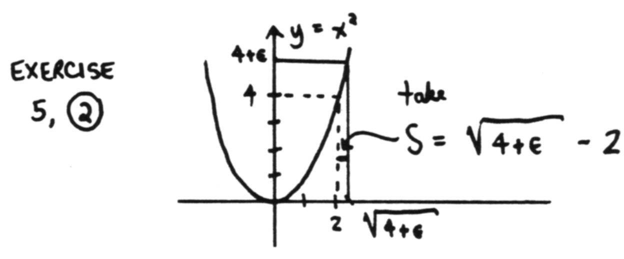


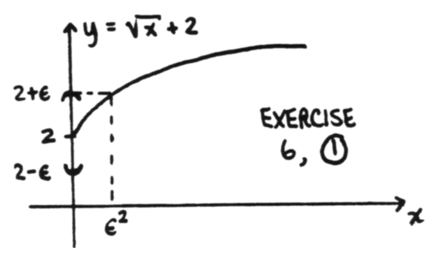
2. Step 1. It must be shown that we can get x^2 as close to 4 as desired, by requiring that x be sufficiently close to 2.

   Step 2. Let $\epsilon > 0$. We want to get x^2 in the interval $(4 - \epsilon, 4 + \epsilon)$.

   Step 3. When $y = 4 - \epsilon$, the corresponding input that lies near 2 is $x = \sqrt{4-\epsilon}$. When $y = 4 + \epsilon$, the corresponding input that lies near 2 is $x = \sqrt{4+\epsilon}$.

   Step 4. The curve $y = x^2$ rises more steeply to the right of 2. Thus, take $\delta := \sqrt{4+\epsilon} - 2$, since this is the shorter distance. Then, whenever x is within δ of 2, x^2 is within ϵ of 4.





EXERCISE 6.

1. Step 1. Define $f(x) := \sqrt{x} + 2$. It must be shown that we can get $f(x)$ as close to 2 as desired, by requiring that x be *in the domain of* f, and sufficiently close to 0.

   Step 2. Let $\epsilon > 0$. We must get $\sqrt{x} + 2$ in the interval $(2 - \epsilon, 2 + \epsilon)$.

   Step 3. When $y = 2+\epsilon$, we have $x = ((2+\epsilon) - 2)^2 = \epsilon^2$. There is no input corresponding to the output $y = 2 - \epsilon$.

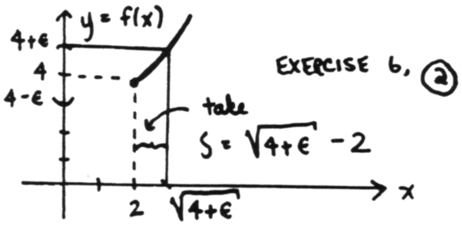
   Step 4. Referring to the sketch, we see that whenever x is within ϵ^2 of 0, and x is in the domain of f, then $f(x)$ is within ϵ of 2. So, take $\delta := \epsilon^2$.

2. Step 1. It must be shown that we can get $f(x)$ as close to 4 as desired, by requiring that x be *in the domain of* f, and sufficiently close to 2.

   Step 2. Let $\epsilon > 0$. We must get $f(x)$ in the interval $(4-\epsilon, 4+\epsilon)$.

   Step 3. When $y = 4+\epsilon$, we have $x = \sqrt{4+\epsilon}$. There is no input corresponding to the output $y = 4-\epsilon$.

   Step 4. Take $\delta := \sqrt{4+\epsilon} - 2$. Then, whenever x is within δ of 2, AND x is in the domain of f, then $f(x)$ is within ϵ of 4.

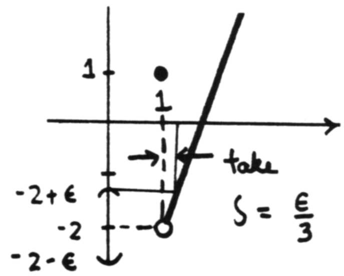


EXERCISE 7.

1. The subsentence '$|x| < \delta$' says that x must be within δ of 0.
2. The subsentence '$0 < |x|$' says that x must not equal 0.

EXERCISE 8.

1. The graph of f is shown below.



2. $\mathcal{D}(f) = [1, \infty)$; the function f IS defined at $x = 1$
3. Step 1. It must be shown that we can get $f(x)$ as close to -2 as desired, by requiring that x be *in the domain of* f and sufficiently close to 1, but not equal to 1.

   Step 2. Let $\epsilon > 0$. We must get $f(x)$ in the interval $(-2-\epsilon, -2+\epsilon)$.

   Step 3. When $y = -2+\epsilon$, we have $x = \frac{((-2+\epsilon)+5)}{3} = 1 + \frac{\epsilon}{3}$. There is no input corresponding to the output $y = -2-\epsilon$.

   Step 4. Take $\delta := \frac{\epsilon}{3}$. Then, whenever $0 < |x-1| < \delta$ and $x \in \mathcal{D}(f)$, then $|f(x) - (-2)| < \epsilon$.
4. For this example, $c = 1$ and $\delta = \frac{\epsilon}{2}$. For the given function f, the sentence

$$0 < |x-1| < \frac{\epsilon}{3} \quad \text{and} \quad x \in \mathcal{D}(f)$$

   is true precisely when $x \in (1, 1 + \frac{\epsilon}{3})$.

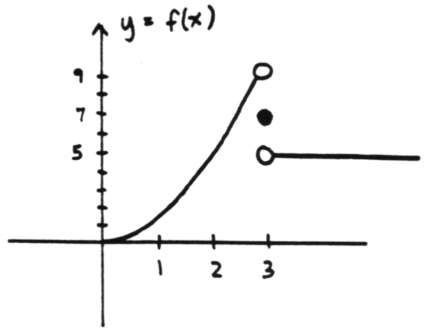
EXERCISE 9.

1. **DEFINITION.** Let f be a function that is defined at least on an interval of the form (a, c), where $a < c$. Then:

$$\lim_{x \to c^-} f(x) = l \iff \begin{array}{l} \text{for every } \epsilon > 0, \text{ there exists } \delta > 0, \text{ such that if} \\ x \in (c-\delta, c), \text{ then } |f(x) - l| < \epsilon \end{array}$$

   There are other correct ways to phrase this definition.

2. The graph of f is shown below.



3. $\lim_{x\to 3^+} f(x) = 5$

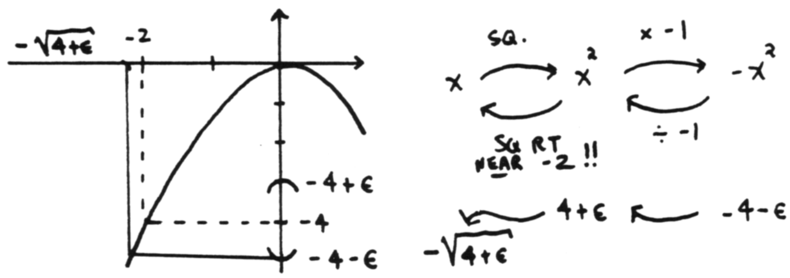
   $\lim_{x\to 3^-} f(x) = 9$

4. The two-sided limit does not exist, because it is impossible to get $f(x)$ simultaneously close to both 9 and 5 .
5. If f is redefined so that $f(x) = 9$ for $x > 3$, then both one-sided limits will equal 9, and the two-sided limit will also exist and equal 9 .

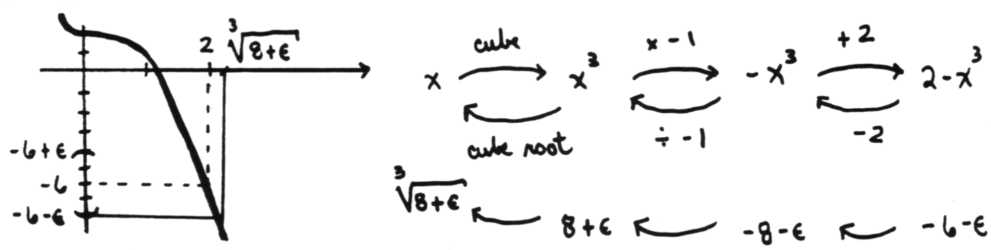
END-OF-SECTION EXERCISES:

In questions 1–4, the 4-step process is summarized via the given sketch, and the required δ is given.

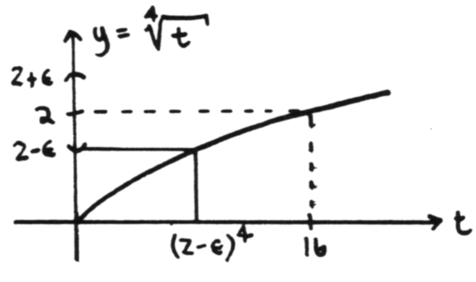
1. When 'undoing' the output $-4-\epsilon$, it is important to take the input that lies near -2! Take $\delta := -2 + \sqrt{4+\epsilon}$.



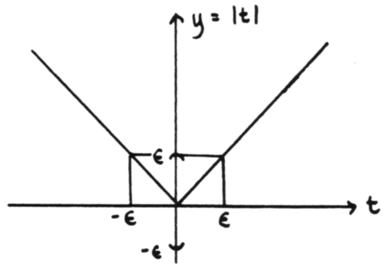
2. Take: $\delta := \sqrt[3]{8+\epsilon} - 2$



3. Take $\delta := 16 - (2-\epsilon)^4$, since this is the shorter distance.



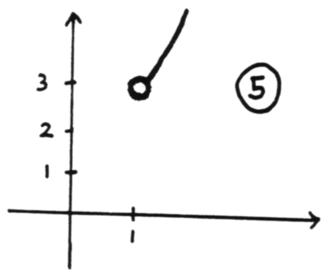
4. Take: $\delta := \epsilon$



5. $\lim_{x\to 1} f(x) = 3$

   $\lim_{x\to 1^+} f(x) = 3$

   $\lim_{x\to 1^-} f(x)$ is not defined, since f is not defined to the left of 1



6. $\lim_{x\to 1} f(x) = 3$

   $\lim_{x\to 1^+} f(x)$ is not defined, since f is not defined to the right of 1

   $\lim_{x\to 1^-} f(x) = 3$

7. $\lim_{x\to -1} g(x) = -1$

   $\lim_{x\to -1^+} g(x) = -1$

   $\lim_{x\to -1^-} g(x) = -1$

8. $\lim_{x\to 2} g(x)$ does not exist

   $\lim_{x\to 2^+} g(x) = 3$

   $\lim_{x\to 2^-} g(x) = 1$

9. TRUE! Indeed, if $\lim_{x\to c} f(x) = l$ and f is defined on both sides of c, then both one-sided limits must also exist and equal l.

10. FALSE! (See, for example, problem #8.) However, if both one-sided limits exist and *are equal*, then the two-sided limit exists and has the same value.