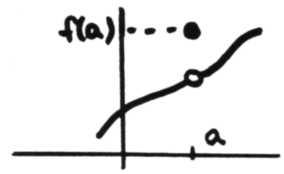
CHAPTER 7. THE DEFINITE INTEGRAL

Section 7.1 Using Antiderivatives to find Area

Quick Quiz:

1. The Max-Min Theorem guarantees numbers $m \in [x, x+h]$ and $M \in [x, x+h]$ for which $f(m)$ is the minimum value of f on $[x, x+h]$, and $f(M)$ is the maximum value of f on $[x, x+h]$.
2. If f is continuous at a, then as $x \to a$, it must be that $f(x) \to f(a)$.
3. Any sketch where f IS defined at a, but f is NOT continuous at a, will work!



4. $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$. Then, the desired area is given by: $F(2) - F(0) = 2^3 - 0^3 = 8$
5. The desired area is given by: $F(d) - F(c)$

END-OF-SECTION EXERCISES:

1. 

   approximation by a triangle: $\frac{1}{2}(1)(e-1) \approx 0.86$

   actual area: Using integration by parts, an antiderivative of $f(x) = \ln x$ is $F(x) = x \ln x - x$. Then:

$$F(e) - F(1) = (e \ln e - e) - (1 \ln 1 - 1) = (e - e) - (0 - 1) = 1$$

3. 

   approximation by a trapezoid: $\frac{1}{2}(4-1)(1+2) = \frac{1}{2}(9) = \frac{9}{2} = 4.5$

   actual area: An antiderivative of $f(x) = \sqrt{x} = x^{1/2}$ is $F(x) = \frac{2}{3}x^{3/2} = \frac{2}{3}\sqrt{x^3}$. Then:

$$F(4) - F(1) = \frac{2}{3}\sqrt{4^3} - \frac{2}{3}\sqrt{1^3} = \frac{2}{3}(8) - \frac{2}{3}(1) = \frac{2}{3}(7) = \frac{14}{3} \approx 4.67$$

Section 7.2 The Definite Integral

Quick Quiz:

1. The indefinite integral $\int f(x)\,dx$ gives all the antiderivatives of the function f; by the Fundamental Theorem of Integral Calculus, if just *one* of these antiderivatives is known, then the definite integral $\int_a^b f(x)\,dx$ can be computed!
2. See page 409.
3. The notation $F(x)\big|_a^b$ means $F(b) - F(a)$.
4. $\int_{-1}^{2} x^2\,dx = \frac{x^3}{3}\big|_{-1}^{2} = \frac{1}{3}(2^3 - (-1)^3) = \frac{1}{3}(8 - (-1)) = \frac{1}{3}(9) = 3$
5. $\int_{-1}^{1} x^3\,dx = \frac{x^4}{4}\big|_{-1}^{1} = \frac{1}{4}(1^4 - (-1)^4) = 0$. On the interval $[-1, 1]$, there is the same amount of area *above* the graph of $y = x^3$, as there is *below*.

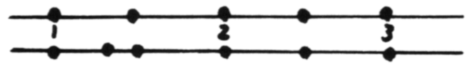
END-OF-SECTION EXERCISES:

1. $\frac{48}{5}$
3. -6
5. $\frac{1}{3}\ln 2$
7. $1 + e^2$
9. The desired area is: $\frac{19}{24} + \frac{81}{8} = \frac{131}{12}$

Section 7.3 The Definite Integral as the Limit of Riemann Sums

Quick Quiz:

1. A *partition* of an interval $[a,b]$ is a finite set of points from $[a,b]$ that includes both a and b.
2. The length of the *longest* subinterval must be $\frac{1}{2}$:
   $P_1 = \{1, 1.5, 2, 2.5, 3\}$
   $P_2 = \{1, 1.3, 1.5, 2, 2.5, 3\}$



3. There is NOT a unique Riemann sum for f corresponding to this partition; any number x_1^* may be chosen from the subinterval $[0,1)$; any number x_2^* may be chosen from the second subinterval $[1,2)$, etc.
4. Think of a rectangle with 'width' dx and 'height' $f(x)$, where x is a number between a and b.

END-OF-SECTION EXERCISES:

1. EXP
3. SENTENCE; TRUE
5. SENTENCE; TRUE
7. SENTENCE; TRUE
9. SENTENCE; TRUE

Section 7.4 The Substitution Technique applied to Definite Integrals

Quick Quiz:

1. $\int (2x-1)^3\,dx = \frac{1}{2}\int u^3\,du = \frac{1}{2}\frac{u^4}{4} + C = \frac{1}{8}(2x-1)^4 + C$;
   $\int_0^{1/2}(2x-1)^3\,dx = \frac{1}{8}(2x-1)^4\,\big|_0^{1/2} = \frac{1}{8}(0-1) = -\frac{1}{8}$

   $u = 2x-1$
   $du = 2dx$

2. $\int_0^{1/2}(2x-1)^3\,dx = \frac{1}{2}\int_0^{1/2}(2x-1)^3\,2\,dx = \frac{1}{2}\int_{-1}^{0} u^3\,du = \frac{1}{2}\frac{u^4}{4}\,\big|_{-1}^{0} = \frac{1}{8}(0-1) = -\frac{1}{8}$

   $x=0 \Rightarrow u=-1$
   $x=\frac{1}{2} \Rightarrow u=0$

3. $u = \ln x \quad dv = dx$
   $du = \frac{1}{x}dx \quad v = x$

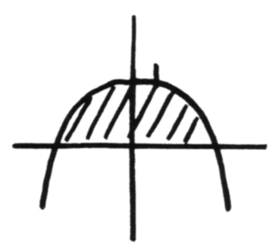
$$\begin{aligned}\int_1^e \ln x\,dx &= x\ln x\,\big|_1^e - \int_1^e x\cdot\frac{1}{x}\,dx \\ &= (e\ln e - 1\ln 1) - x\,\big|_1^e \\ &= e - (e-1) = 1\end{aligned}$$

END-OF-SECTION EXERCISES:

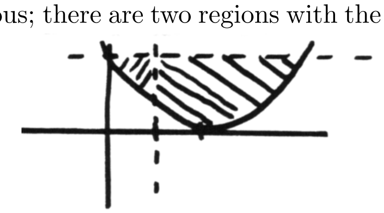
1. 0
3. ≈ 0.024
5. ≈ 1.931

Section 7.5 The Area Between Two Curves

Quick Quiz:

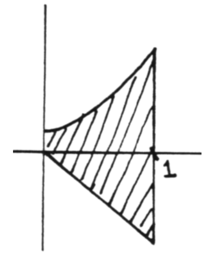


1. $\int_c^d \big(g(x) - f(x)\big)\,dx$
2. The x-axis is described by $y=0$. The intersection points are found by:
$$-x^2 + 1 = 0 \iff x^2 = 1 \iff x = \pm 1$$
   Using symmetry, the desired area is:
$$2\int_0^1 (-x^2+1)\,dx = (x - \frac{x^3}{3})\,\big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$
3. A quick sketch shows that the phrase IS ambiguous; there are two regions with the indicated boundaries. Which is desired? Or, are both desired?



4.

$$\int_0^1 (e^x - (-x))\,dx = \int_0^1 (e^x + x)\,dx$$
$$= (e^x + \frac{x^2}{2})\,\Big|_0^1$$
$$= (e + \frac{1}{2}) - e^0 = e + \frac{1}{2} - 1 = e - \frac{1}{2}$$



END-OF-SECTION EXERCISES:

1. $\dfrac{2}{15}$

3. $\dfrac{32}{3}$

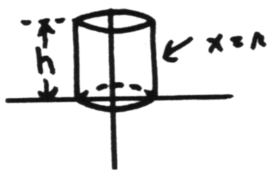
5. ≈ 2.438

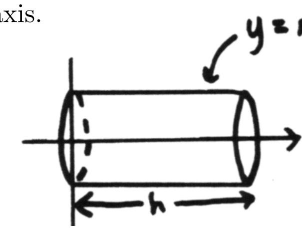
7. $20\dfrac{1}{4}$

Section 7.6 Finding the Volume of a Solid of Revolution—Disks

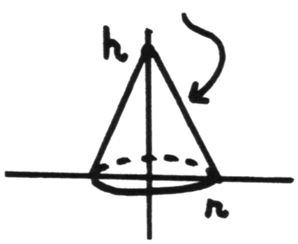
Quick Quiz:

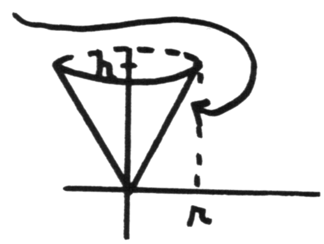
1. Revolve $x = r$ about the y-axis; or revolve $y = r$ about the x-axis.



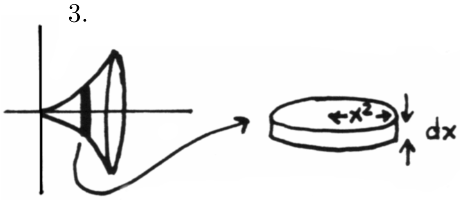


2. Revolve $y = -\frac{h}{r}x + h$ about the y-axis; or revolve $y = \frac{h}{r}x$ about the y-axis. (There are other correct answers.)





3.



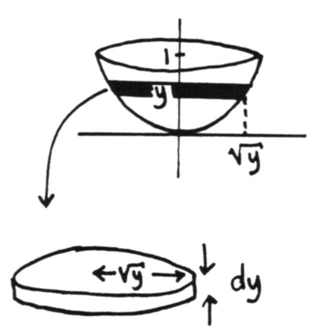
$$\int_0^1 \pi(x^2)^2\,dx = \pi\frac{x^5}{5}\,\Big|_0^1 = \frac{\pi}{5}(1-0) = \frac{\pi}{5}$$

4. intersection points of $y = x^2$ and $y = 1$: $x^2 = 1 \iff x = \pm 1$

Also: $y = x^2 \iff x = \pm\sqrt{y}$

A typical 'slice' at a distance y has volume $\pi(\sqrt{y})^2\,dy$. The desired volume is:

$$\int_0^1 \pi(\sqrt{y})^2\,dy = \int_0^1 \pi y\,dy = \pi\frac{y^2}{2}\,\Big|_0^1 = \frac{\pi}{2}(1-0) = \frac{\pi}{2}$$



END-OF-SECTION EXERCISES:

1. $\dfrac{4\pi}{3}$

3. $\dfrac{\pi}{2}$

5. 8π

7. $\dfrac{128\pi}{5}$

9. $\dfrac{8\pi}{3}$

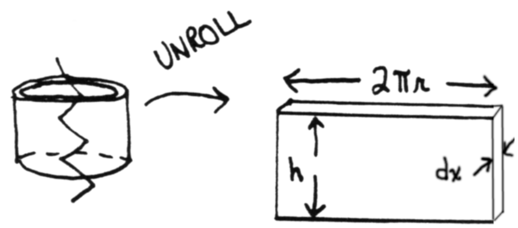
11. $\dfrac{\pi}{4}$

Section 7.7 Finding the Volume of a Solid of Revolution—Shells

Quick Quiz:

1. 'Cut' the shell and unroll it; the volume is:

$$2\pi r \cdot h \cdot dx$$



2.
$$\int_0^2 2\pi x(x)\,dx = 2\pi\frac{x^3}{3}\Big|_0^2 = \frac{2\pi}{3}(8-0) = \frac{16\pi}{3}$$

3. To use horizontal disks would require disks 'with holes'. Thus, in this case, shells are easier to use.

END-OF-SECTION EXERCISES:

1. $\dfrac{4\pi}{3}$

3. 2π