CHAPTER 4. THE DERIVATIVE

Section 4.1 Tangent Lines

Quick Quiz:

1. Let $f(x) = x$. Then:

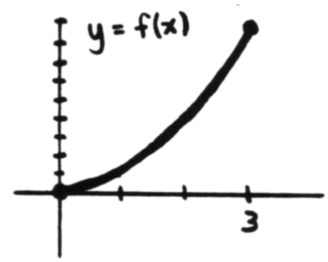
$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{(2+h)-2}{h} = \lim_{h\to 0}\frac{h}{h} = 1$$

Thus, as expected, the slope of hte tangent line to f at the point $(2,2)$ is 1 .

2. The dummy variable is h . Using the dummy variable t, the limit can be rewritten as:

$$\lim_{t\to 0}\frac{f(x+t)-f(x)}{t}$$

3. In the limit, x represents the x-value of a point where the slope of the tangent line is desired.
4. In the limit, the difference quotient $\frac{f(x+h)-f(x)}{h}$ represents the slope of a secant line through the points $(x, f(x))$ and $(x+h, f(x+h))$. This secant line is being used as an approximation to the tangent line at the point $(x, f(x))$.
5. The function f is graphed below. Since f is only defined to the right of 0, the limit is actually a right-hand limit:



$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+}\frac{h^2-0}{h} = \lim_{h\to 0^+} h = 0$$

The slope of the tangent line at the point $(0,0)$ is 0 .

End-of-Section Exercises:

1. EXP
3. SEN; CONDITIONAL
5. SEN; TRUE
7. $g(0.1) = \frac{f(x+0.1)-f(x)}{0.1}$; $g(\Delta x) = \frac{f(x+\Delta x)-f(x)}{\Delta x}$
9. $h \in \mathcal{D}(g) \iff \big(h \neq 0 \text{ and } x+h \in \mathcal{D}(f)\big)$
11. When $\lim_{h\to 0} g(h)$ exists, it tells the slope of the tangent line to the graph of f at the point $(x, f(x))$.

Section 4.2 The Derivative

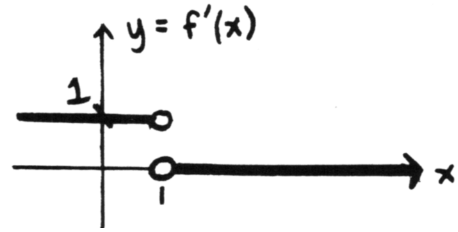
Quick Quiz:

1. When the limit exists:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

2. f' is the derivative *function*; $f'(x)$ is a particular output of this function, when the input is x.
3. $A - B = (0,2)\cup(2,4)$; $B - A = \{4\}$

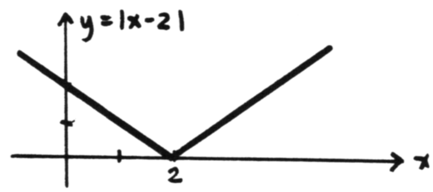
4. $\mathcal{D}(f') = \mathbb{R} - \{1\}$; its graph is:



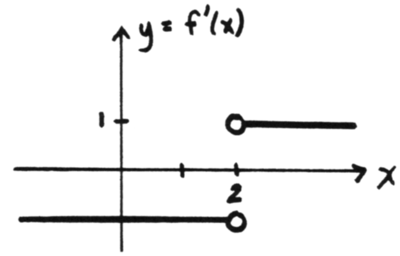
5. TRUE. If the limit $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ exists, then, in particular, f must be defined at x (so that $f(x)$ makes sense).

End-of-Section Exercises:

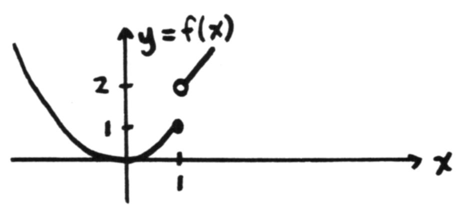
1. The graph of f is shown below. Here, $\mathcal{D}(f) = \mathbb{R}$.



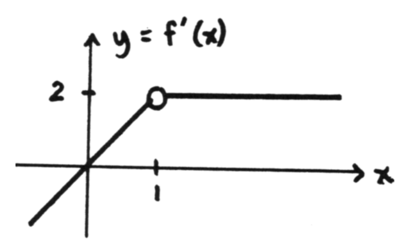
When $x > 2$, the slopes of the tangent lines equal 1.
When $x < 2$, the slopes of the tangent lines equal -1.
There is no tangent line at $x = 2$.
The graph of f' is shown at right. Here, $\mathcal{D}(f') = \mathbb{R} - \{2\}$.



3. The graph of f is shown below. Here, $\mathcal{D}(f) = \mathbb{R}$.



When $x > 1$, the slopes of the tangent lines equal 2.
When $x < 1$, the slopes of the tangent lines equal $2x$ (as per an example in the text).
There is no tangent line at $x = 1$.
The graph of f' is shown at right. Here, $\mathcal{D}(f') = \mathbb{R} - \{1\}$.



5. Note that $f(2) = \frac{1}{2-1} = 1$. Then:

$$\begin{aligned}\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0} \frac{\frac{1}{(2+h)-1} - 1}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{1+h} - \frac{1+h}{1+h}}{h} \\ &= \lim_{h\to 0} \frac{1-(1+h)}{h(1+h)} \\ &= \lim_{h\to 0} \frac{-1}{1+h} = -1\end{aligned}$$

Thus, $f'(2) = -1$. That is, the slope of the tangent line to the graph of f at the point $(2, 1)$ is -1 .

7. $y - 9 = 6(x - 3)$
9. $y = 1$

Section 4.3 Some Very Basic Differentiation Formulas

Quick Quiz:

1. $f(x) = x^{1/2}$; $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$. In Leibniz notation: $\frac{df}{dx} = \frac{1}{2\sqrt{x}}$
2. TRUE. The derivative of a constant equals zero.
3. $y' = 3x^2$; the slope of the tangent line at $x = 2$ is $y'(2) = 3(2^2) = 12$. TRUE.
4.

$$\begin{aligned}(a-b)^4 = (a+(-b))^4 &= (1)a^4 + (4)a^3(-b) + (6)a^2(-b)^2 + (4)a(-b)^3 + (1)(-b)^4 \\ &= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4\end{aligned}$$

1
1 2 1
1 3 3 1
1 4 6 4 1

5. $g'(x) = e^x + \frac{1}{x}$

End-of-Section Exercises:

1. Multiply out, differentiate term-by-term, and simplify: $f'(x) = 6(2x + 1)^2$
3.

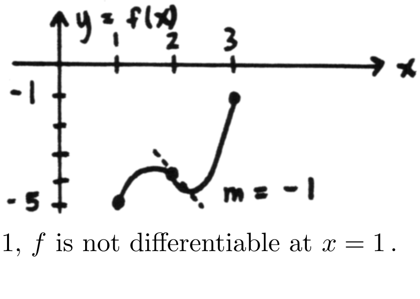
$$h'(x) = \begin{cases} 6x - 2 & \text{for } x \ge 1 \\ 4 & \text{for } x < 1 \end{cases}$$

$\mathcal{D}(h) = \mathcal{D}(h') = \mathbb{R}$

Section 4.4 Instantaneous Rates of Change

Quick Quiz:

1. $\frac{f(2)-f(1)}{2-1} = \frac{2^3-1^3}{1} = 8 - 1 = 7$; this number represents the slope of the secant line through the points $(1, 1^3)$ and $(2, 2^3)$
2. $f'(x) = 3x^2$; $f'(1) = 3(1) = 3$. This number represents the slope of the tangent line at the point $(1, 1^3)$.
3. less than; once we move to the right of $x = 1$, the rates of change increase
4. One correct sketch is given:

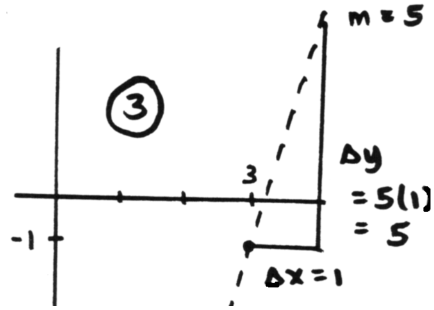


5. Since f is not continuous at $x = 1$, f is not differentiable at $x = 1$.

End-of-Section Exercises:

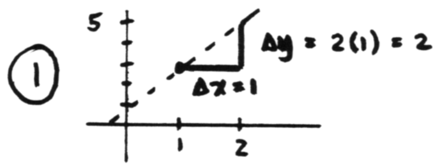
In all cases, the 'predicted value' for $f(x_2)$ from known information at x_1 is given by

$$f(x_2) \approx f(x_1) + (\Delta x)(f'(x_1)) ,$$



where $\Delta x_2 = x_2 - x_1$.

1. Here, $\Delta x = 2 - 1 = 1$; $f(2) \approx 3 + (1)(2) = 5$
3. Here, $\Delta x = 4 - 3 = 1$; $f(4) \approx -1 + (1)(5) = 4$



Section 4.5 The Chain Rule (Differentiating Composite Functions

Quick Quiz:

1. See page 231. The Chain Rule tells us how to differentiate composite functions.
2. $f'(x) = 7\sqrt{2}(1-x)^6(-1) = -7\sqrt{2}(1-x)^6$
3. $\frac{dy}{dt} = \frac{dy}{dw} \cdot \frac{dw}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dt}$
4. ... tells us that to find out how fast $f \circ g$ changes with respect to x, we find out how fast f changes with respect to $g(x)$, and multiply by how fast g changes with respect to x
5. $f(x) = \frac{1}{3}\ln(2x+1)$, $f'(x) = \frac{1}{3} \cdot \frac{1}{2x+1} \cdot 2 = \frac{2}{3(2x+1)}$

End-of-Section Exercises:

1. $f'(x) = \dfrac{-e^x}{\sqrt{(e^x-1)^3}} + 1$
3. $\dfrac{dy}{dx} = 3e^{3x}$
5. $y' = 33(3t-4)^{10}$
7. $g'(t) = \dfrac{2t+1}{2\sqrt[6]{(t^2+2t+1)^5}}$
9. $f'(y) = -7e^{-y} + \dfrac{1}{y}$
11. $\dfrac{dy}{dx} = \dfrac{3}{x}(\ln x)^2$
13. $\dfrac{dy}{dt} = \dfrac{2\sqrt{t-1}+1}{2\sqrt{t-1}(t+\sqrt{t-1})^2}$

Section 4.6 Differentiating Products and Quotients

Quick Quiz:

1. See page 239.
2. See page 244.
3. $f'(x) = x \cdot 5(x+1)^4(1) + (1)(x+1)^5$
4. Using the Quotient Rule:

$$\begin{aligned} f'(x) &= \frac{e^{2x}(2) - (2x+1)\cdot 2e^{2x}}{(e^{2x})^2} \\ &= \frac{2e^{2x}(1-(2x+1))}{e^{4x}} \\ &= \frac{-4xe^{2x}}{e^{4x}} \end{aligned}$$

5. Using a 'generalized' product rule:

$$y' = (1)(x+1)(x^2+1) + x(1)(x^2+1) + x(x+1)(2x)$$

End-of-Section Exercises:

1.
$$y' = 2(2-x)^2(1-2x)$$
$$y(0) = 0\,,\quad y(t^2) = t^2(2-t^2)^3$$
$$y'(0) = 8\,,\quad y'(t) = 2(2-t)^2(1-2t)$$

3.
$$f'(x) = \mathrm{e}^x(\frac{1}{x} + \ln x)$$
$$\mathcal{D}(f) = (0,\infty)\,,\quad \mathcal{D}(f') = (0,\infty)$$
$$f'(\mathrm{e}^x) = \mathrm{e}^{(\mathrm{e}^x)}(\frac{1}{\mathrm{e}^x} + x)\,,\quad f'(\mathrm{e}^2) = \mathrm{e}^{(\mathrm{e}^2)}(\frac{1}{\mathrm{e}^2} + 2)$$

5.
$$g'(x) = \mathrm{e}^{x+\mathrm{e}^x}$$
$$\lim_{x\to 0} g(x) = \mathrm{e}\,,\quad \lim_{x\to 0} g'(x) = \mathrm{e}$$
$$\mathcal{D}(g) = \mathbb{R}\,,\quad g(g'(g(0))) = \mathrm{e}^{\mathrm{e}^{(\mathrm{e}^{\mathrm{e}+\mathrm{e}^{\mathrm{e}}})}}$$

7. $h'(x) = \frac{x}{x+1}$; the tangent line is horizontal, and has equation $y = 0$
9. $f'(x) = 4\mathrm{e}^{2x}(2x+1)^6(x+4)$; the tangent line has equation $y = 16x + 1$
11. $h(t) = \frac{-12\mathrm{e}}{(3t-1)^5}$; the tangent line has equation $y - \mathrm{e} = -12\mathrm{e}(t - \frac{2}{3})$
13.
$$y' = 0 \iff \big(x = 3 \text{ or } x = -1 \text{ or } x = \frac{1}{2} \text{ or } x = \frac{3 \pm \sqrt{17}}{2}\big)$$

Section 4.7 Higher Order Derivatives

Quick Quiz:

1. The 'higher derivatives' of a function f are the derivatives of the form $f^{(n)}$ for $n \geq 2$. That is, the second derivative, third derivative, fourth derivative, etc., are the 'higher derivatives' of f.
2. prime notation: $f''(x)$

   Leibniz notation: $\dfrac{d^2 f}{dx^2}(x)$
3. $\sum_{i=1}^{3} i^{i+1} = 1^{1+1} + 2^{2+1} + 3^{3+1} = 1 + 8 + 81 = 90$
4.
$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot \frac{5!}{5!} = \frac{10!}{5!}$$
5. $\dfrac{d}{dx} \displaystyle\sum_{i=1}^{n} f_i(x) = \sum_{i=1}^{n} {f_i}'(x)$

End-of-Section Exercises:

1. SEN; TRUE
3. EXP

5. SEN; CONDITIONAL
7. SEN; TRUE
9. EXP
11. EXP
13. EXP
15. SEN; TRUE
17. SEN; CONDITIONAL

Section 4.8 Implicit Differentiation (Optional)

Quick Quiz:

1.
$$\begin{aligned}\frac{d}{dx}(xy^2) &= \frac{d}{dx}(2)\\ x(2y^1)\frac{dy}{dx} + (1)y^2 &= 0\\ \frac{dy}{dx} &= \frac{-y^2}{2xy}\end{aligned}$$

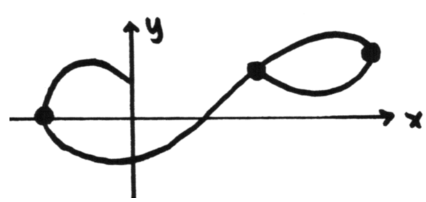
2. For $x > 0$:
$$\begin{aligned}\ln y &= \ln(x^{2x}) = 2x\ln x\\ \frac{1}{y}\frac{dy}{dx} &= (2x)\frac{1}{x} + (2)\ln x = 2 + 2\ln x = 2(1+\ln x)\\ \frac{dy}{dx} &= y\cdot 2(1+\ln x) = 2x^{2x}(1+\ln x)\end{aligned}$$

3. Put the equation in standard form, by completing the square:
$$\begin{aligned}x^2 - 2x + y^2 = 8 &\iff (x^2 - 2x + (\tfrac{-2}{2})^2) + y^2 = 8 + 1\\ &\iff (x-1)^2 + (y-0)^2 = 3^2\end{aligned}$$

The equation graphs as the circle centered at $(1, 0)$ with radius 3.

4. There are *many* possible correct answers. Here are two:
y given explicitly in terms of x: $y = x^2 + 2x + 1$
y given implicitly in terms of x: $xy^2 = x + y$

5.



End-of-Section Exercises:

1. The graph is the circle centered at $(-2, 1)$ with radius 1.
y is NOT locally a function of x at the points $(-1, 1)$ and $(-3, 1)$. (There are vertical tangent lines here.)
The equation of the tangent line at the point $(-2, 2)$ is $y = 2$.
The equation of the tangent line at the point $(-1, 1)$ is $x = -1$.

3. The graph is the circle centered at $(-2, 1)$ with radius 2.
y is NOT locally a function of x at the points $(0, 1)$ and $(-4, 1)$; there are vertical tangent lines here.
The equation of the tangent line at the point $(-1, 1 + \sqrt{3})$ is:
$$y - (1 + \sqrt{3}) = -\frac{1}{\sqrt{3}}(x - (-1))$$

Section 4.9 The Mean Value Theorem

Quick Quiz:

1. See page 266.
2. The word ‘mean’ refers to ‘average’; the Mean Value Theorem guarantees (under certain hypotheses) a place in an interval (a, b) where the *instantaneous* rate of change is the same as the *average* rate of change over the entire interval.
3. The average rate of change of f on the interval $[1, 3]$ is:

$$\frac{f(3) - f(1)}{3 - 1} = \frac{27 - 1}{2} = 13$$

The instantaneous rates of change are given by $f'(x) = 3x^2$. We esek $c \in (1, 3)$ for which $f'(c) = 13$:

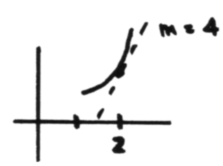
$$\begin{aligned} f'(c) = 13 &\iff 3c^2 = 13 \\ &\iff c^2 = \frac{13}{3} \\ &\iff c = \pm\sqrt{\frac{13}{3}} \end{aligned}$$

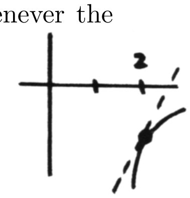
Choosing the value of c in the desired interval, we get $c = \sqrt{\frac{13}{3}}$.

4. If f WERE continuous on $[a, b]$, then there would have to be (by the MVT) a number $c \in (a, b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$. Thus, it must be that f is NOT continuous on $[a, b]$; that is, f ‘goes bad’ at (at least one) endpoint.
5. If f WERE differentiable on (a, b), then the MVT would guarantee that there must be $c \in (a, b)$ with $f'(c)$ equal to the average rate of change of f over $[a, b]$. Therefore, we can conclude that f is NOT differentiable on (a, b). That is, there is at least one value of x in the interval (a, b) where $f'(x)$ does not exist.

End-of-Section Exercises:

1. The limit gives the slope of the tangent line to the graph of f at the point $(x, f(x))$, whenever the tangent line exists and is non-vertical.
3. There is a tangent line to the graph of f when $x = 2$, and its slope is 4.

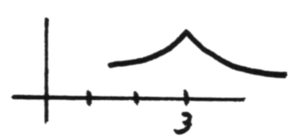




5. Let $f(x) = -x^2$. Then:

$$\begin{aligned} f'(x) &:= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{-(x+h)^2 - (-x^2)}{h} \\ &= \lim_{h\to 0} \frac{-(x^2 + 2xh + h^2) + x^2}{h} = \lim_{h\to 0} \frac{h(-2x - h)}{h} \\ &= \lim_{h\to 0} (-2x - h) = -2x \end{aligned}$$

7. Put a ‘kink’ in the graph when $x - 3$.



9.

$$\begin{gathered} f'(x) = e^{2x} \ln(2 - x) + 2xe^{2x} \ln(2 - x) - \frac{xe^{2x}}{2 - x} \\ \mathcal{D}(f) = (-\infty, 2)\,, \quad \mathcal{D}(f') = (-\infty, 2) \end{gathered}$$

The tangent line when $x = 0$ has equation $y = (\ln 2)x$.