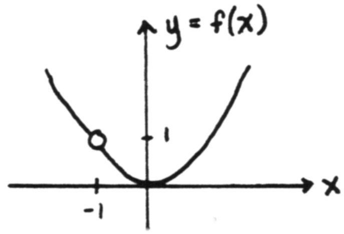
CHAPTER 3. LIMITS AND CONTINUITY

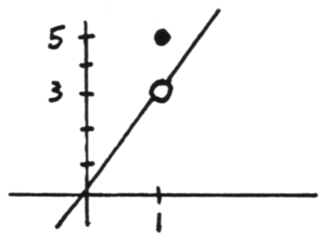
Section 3.1 Limits—The Idea

Quick Quiz:

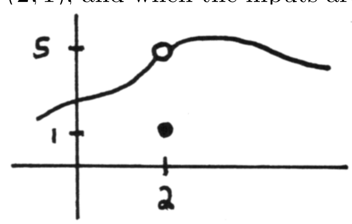
1. $\lim_{x\to -2} x^3 = (-2)^3 = -8$
2. $\lim_{x\to -1} f(x) = (-1)^2 = 1$



3. $\lim_{x\to 1} f(x) = 3(1) = 3$



4. There are many correct graphs. The graph must contain the point $(2, 1)$; and when the inputs are close to 2 (but not equal to 2), the outputs must be close to 5.



5. $|t - (-1)| \le 4$

End-of-Section Exercises:

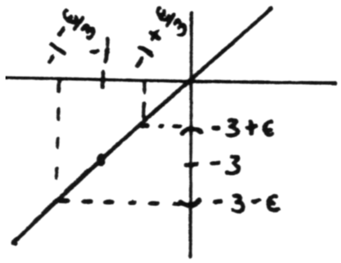
1. EXP
3. SEN; T
5. SEN; C
7. SEN; T
9. SEN; C
11. EXP
13. SEN; (always) T
15. SEN; (always) T
17. SEN; C
19. SEN; T

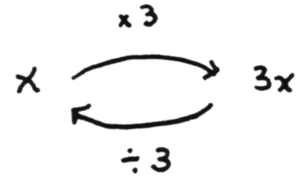
Section 3.2 Limits—Making It Precise

Quick Quiz:

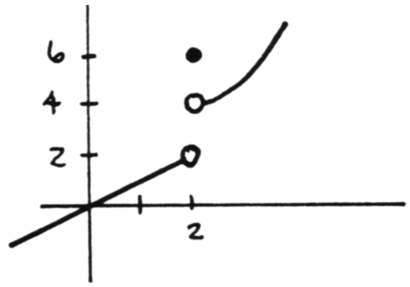
1. $\lim_{x\to c} f(x) = l \iff \forall\ \epsilon > 0,\ \exists\ \delta > 0,$ such that if $0 < |x-c| < \delta$ and $x \in \mathcal{D}(f)$, then $|f(x)-l| < \epsilon$

2. The 'four step process' is summarized using the mapping diagram and and sketch given below. Given $\epsilon > 0$, take $\delta = \frac{\epsilon}{3}$.



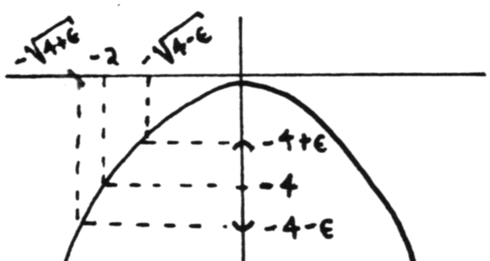


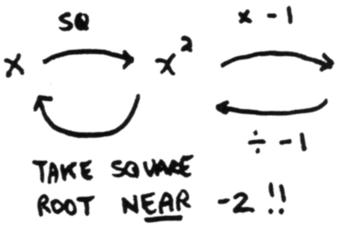
3. $\lim_{x\to 2} f(x)$ does not exist
$\lim_{x\to 2^+} f(x) = 4$
$\lim_{x\to 2^-} f(x) = 2$



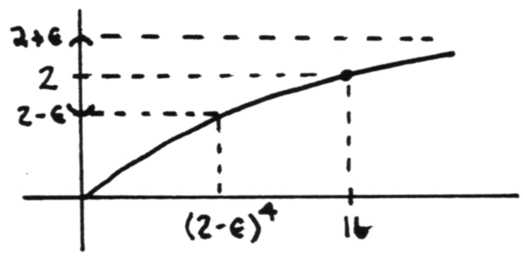
End-of-Section Exercises:

1. When 'undoing' the output $-4-\epsilon$, it is important to take the input that lies near -2! Take: $\delta := -2 + \sqrt{4+\epsilon}$

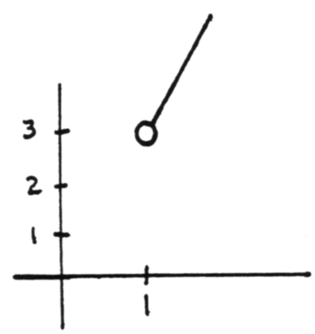




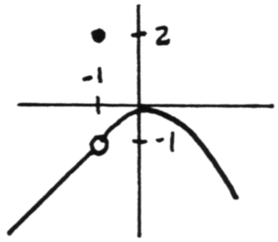
3. Take $\delta := 16 - (2-\epsilon)^4$, since this is the shorter distance.



5. $\lim\limits_{x\to 1} f(x) = 3$
$\lim\limits_{x\to 1^+} f(x) = 3$
$\lim\limits_{x\to 1^-} f(x)$ is not defined, since f is not defined to the left of 1



7. $\lim\limits_{x\to -1} g(x) = -1$
$\lim\limits_{x\to -1^+} g(x) = -1$
$\lim\limits_{x\to -1^-} g(x) = -1$



9. TRUE! Indeed, if $\lim_{x\to c} f(x) = l$ and f is defined on both sides of c, then both one-sided limits must also exist and equal l.

Section 3.3 Properties of Limits

Quick Quiz:

1. To show that an object is unique, a mathematician often supposes that there are TWO, and then shows that they must be equal.
2. As long as both 'component' limits exist, the limit of a sum is the sum of the limits.
3. For all real numbers a and b:
$$|a+b| \le |a| + |b|$$
4. To evaluate the limit, just evaluate the function f at c; that is, substitute the value c into the formula for f.
5. All the component limits exist, so:
$$\lim_{z\to 1} \frac{-2f(z)+g(z)}{h(z)} = \frac{(-2)(3)+5}{2} = -\frac{1}{2}$$

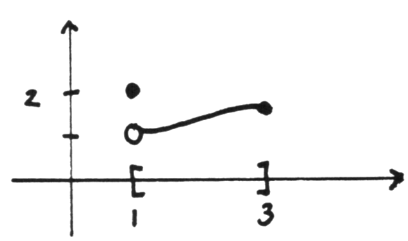
End-of-Section Exercises:

1. SEN; TRUE
3. SEN; TRUE
5. SEN; TRUE
7. SEN; TRUE
9. SEN; FALSE
11. SEN; TRUE
13. SEN; CONDITIONAL
15.
$$\begin{aligned}\lim_{t\to c}[f(t)+g(t)] &= \lim_{t\to c} f(t) + \lim_{t\to c} g(t)\\ &= (-1)+2\\ &= 1\end{aligned}$$
17. There is not enough information to evaluate this limit. We don't know anything about the behavior of f and g, as the inputs approach the number d.

Section 3.4 Continuity

Quick Quiz:

1. A function f is continuous at c if f is defined at c, and $\lim_{x\to c} f(x) = f(c)$.
2. NO! If f were continuous at c, the value of the limit would have to be 3. The discontinuity is removable.
3. f has a nonremovable discontinuity at c if $\lim_{x\to c} f(x)$ does not exist.
4. When f is continuous at c.
5. There are many correct graphs. Here is one:



End-of-Section Exercises:

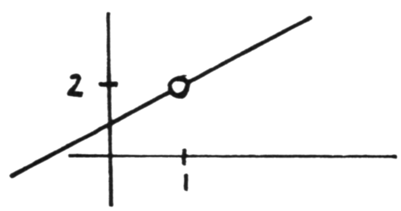
1. SEN; CONDITIONAL
3. EXP
5. SEN; CONDITIONAL
7. SEN; CONDITIONAL
9. SEN; CONDITIONAL
11. SEN; TRUE
13. SEN; FALSE
15. EXP. Out of context, it is not known if this is a POINT (a, b), or an open interval of real numbers. In either case, however, it is an EXPRESSION.

Section 3.5 Indeterminate Forms

Quick Quiz:

1. An 'indeterminate form' is a limit that, upon direct substitution, results in one of the forms: $\frac{0}{0}$, 1^∞, or $\frac{\pm\infty}{\pm\infty}$.
2. $$\begin{aligned}\lim_{x\to 1}\frac{x^2-1}{x-1} &= \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}\\ &= \lim_{x\to 1}(x+1) = 1+1 = 2\end{aligned}$$
3. NO! It is true for all values of x except 1. When x is 1, the left-hand side is not defined; the right-hand side equals 2 .
4. $$y = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} \overset{\text{for } x\neq 1}{=} x+1$$

   The graph is:



5. The graph of f is the same as the graph of $y = \frac{x^2-1}{x-1}$. See (4).
6. $f = g$ if and only if $\mathcal{D}(f) = \mathcal{D}(g)$, and $f(x) = g(x)$ for all x in the common domain.

End-of-Section Exercises:

1. SEN; FALSE
3. SEN; FALSE
5. SEN; TRUE
7. SEN; TRUE. (Either both limits do not exist; or they both exist, and are equal.)
9. $$\begin{aligned}\lim_{x\to -1}\frac{x^3+x^2-3x-3}{x+1} &= \lim_{x\to -1}\frac{(x+1)(x^2-3)}{x+1}\\ &= \lim_{x\to -1} x^2-3\\ &= (-1)^2-3 = -2\end{aligned}$$
11. $$\lim_{x\to 2}\frac{x+2}{x^2+4x+4} = \frac{2+2}{2^2+4(2)+4} = \frac{4}{16} = \frac{1}{4}$$

13. $\lim_{t\to 0^+} (1+t)^{1/t} = e$

Section 3.6 The Intermediate Value Theorem

Quick Quiz:

1. If f is continuous on $[a,b]$, and D is any number between $f(a)$ and $f(b)$, then there exists $c \in (a,b)$ with $f(c) = D$.
2. Since f is continuous on $[1,3]$ and 0 is a number between $f(1)$ and $f(3)$, the Intermediate Value Theorem guarantees the existence of a number c with $f(c) = 0$.
3. TRUE. When the hypothesis of an implication is false, the implication is (vacuously) true.
4. FALSE. Let $x = -1$. Then the hypothesis $|-1| = 1$ is true, but the conclusion $-1 = 1$ is false.
5.

A	B	A ⇒ B
T	T	T
T	F	F
F	T	T
F	F	T

END-OF-SECTION EXERCISES:

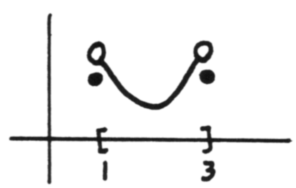
1. TRUE
3. TRUE
5. TRUE
7. TRUE
9. TRUE

Section 3.7 The Max-Min Theorem

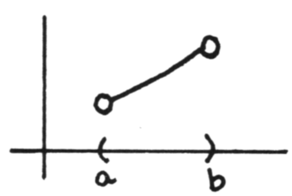
Quick Quiz:

1. The symbol ‘ $\iff$ ’ can also be read as ‘if and only if’.

   The number $f(c)$ is a maximum of f on I if and only if $f(x) \le f(c)$ for all $x \in I$.
2. There are many possible correct graphs.



3. There are many possible correct graphs.

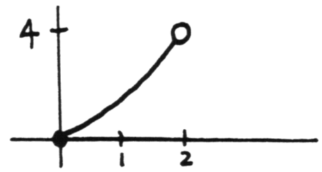


4. If f is continuous on $[a,b]$, then f attains both a maximum and minimum value on $[a,b]$.
5. The contrapositive of the sentence ‘$A \implies B$’ is the sentence ‘not $B \implies$ not A’.

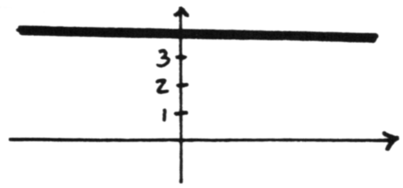
   An implication is equivalent to its contrapositive. That is, the sentences ‘$A \implies B$’ and ‘not $B \implies$ not A’ always have the same truth values, regardless of the truth values of A and B.

END-OF-SECTION EXERCISES:

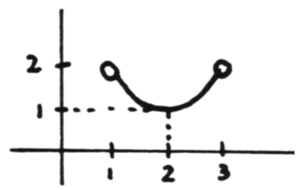
1. The minimum value of f on I is 0; there is no maximum value. The only minimum point is $(0,0)$.



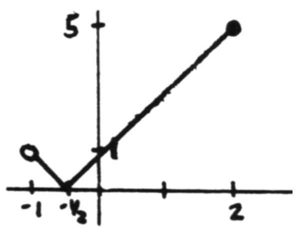
3. The maximum value of f on I is 4; the minimum value of f on I is 4. The points $(x,4)$ for $x \in \mathbb{R}$ are all both maximum and minimum points.



5. The minimum value of f on I is 1; there is no maximum value. The point $(2,1)$ is the only minimum point.



7. The minimum value of f on I is 0; the maximum value is 5. The only minimum point is $(-\frac{1}{2},0)$; the only maximum point is $(2,5)$.

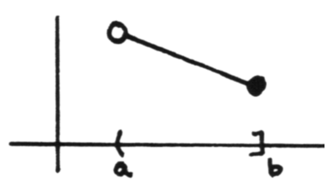


9. TRUE

   Contrapositive: If f does not attain a maximum value on $[a,b]$, then f is not continuous on $[a,b]$.

11. FALSE

   Counterexample: Let f be the function graphed below. Then, the hypothesis 'f is continuous on $(a,b]$' is TRUE, but the conclusion 'f attains a maximum value on $(a,b]$' is FALSE.



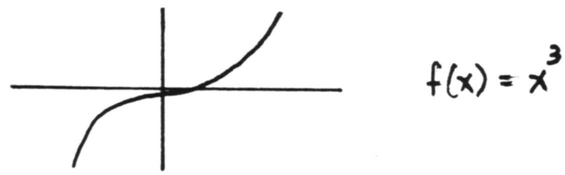
   Contrapositive: If f does not attain a maximum value on $(a,b]$, then f is not continuous on $(a,b]$.

13. TRUE

   Contrapositive: If f does NOT attain both a maximum and minimum value on $[1,2]$, then f is not continuous on $(0,5)$.

15. FALSE.

Counterexample: Let f be the function graphed below. Then the hypothesis 'f is continuous on $\mathbb{R}$' is TRUE, but the conclusion 'f attains a maximum value on $\mathbb{R}$' is FALSE.



Contrapositive: If f does not attain a maximum value on $\mathbb{R}$, then f is not continuous on $\mathbb{R}$.