7.4 The Substitution Technique applied to Definite Integrals

Introduction

Consider the definite integral:

$$\int_0^1 x\sqrt{1-x^2}\,dx$$

To find an antiderivative of $x\sqrt{1-x^2}$ requires a substitution; when this substitution is performed in the context of the definite integral, one must be careful how things are written down.

There are two basic approaches for using substitution in definite integral problems. Both are discussed in this section.

Approach #1

first find an antiderivative;

use it to solve the definite integral

The first approach, which has already been illustrated in an earlier section, is to recognize that *once we have an antiderivative, solving the definite integral problem is easy.* So we can *first solve the corresponding indefinite integral problem,* and then use the simplest antiderivative to compute the desired definite integral.

EXAMPLE

approach #1

Problem: Find $\int_0^1 x\sqrt{1-x^2}\,dx$.

Solution #1: First solve the corresponding indefinite integral problem:

$$\begin{aligned}\int x\sqrt{1-x^2}\,dx &= \frac{1}{-2}\int -2x\sqrt{1-x^2}\,dx \\ &= -\frac{1}{2}\int u^{1/2}\,du \\ &= -\frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)+C \\ &= -\frac{1}{3}(1-x^2)^{3/2}+C \\ &= -\frac{1}{3}\left(\sqrt{1-x^2}\right)^3+C\end{aligned}$$

$$\boxed{\begin{aligned}u &= 1-x^2 \\ du &= -2x\,dx\end{aligned}}$$

The simplest antiderivative is when $C = 0$. Then:

$$\begin{aligned}\int_0^1 x\sqrt{1-x^2}\,dx &= -\frac{1}{3}(\sqrt{1-x^2})^3\Big|_0^1 \\ &= 0-(-\frac{1}{3}\cdot 1) = \frac{1}{3}\end{aligned}$$

EXAMPLE

approach #2;

transform the original definite integral into a NEW definite integral;

changing the limits of integration

Another approach, *that allows the solution to be written down more compactly*, is to transform the original definite integral into a NEW definite integral, as illustrated in this alternate solution:

Solution #2:

$$
\begin{aligned}
\int_0^1 x\sqrt{1-x^2}\,dx &= -\frac{1}{2}\int_0^1 -2x\sqrt{1-x^2}\,dx \\
&= -\frac{1}{2}\int_1^0 u^{1/2}\,du \\
&= -\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_1^0 \\
&= -\frac{1}{3}u^{3/2}\Big|_1^0 \\
&= -\frac{1}{3}(0-1) = \frac{1}{3}
\end{aligned}
$$

NEW limits of integration

$u = 1 - x^2$

$du = -2x\,dx$

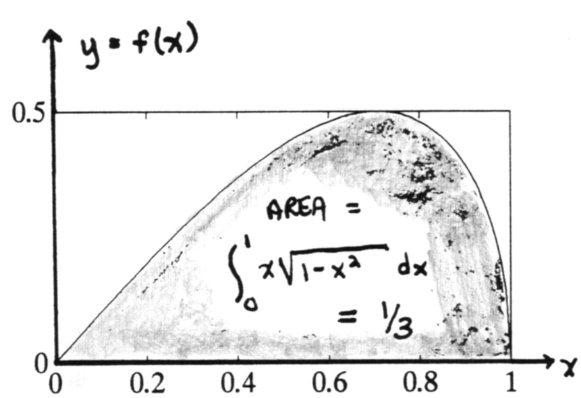
$x = 0 \Rightarrow u = 1 - 0^2 = 1$

$x = 1 \Rightarrow u = 1 - 1^2 = 0$

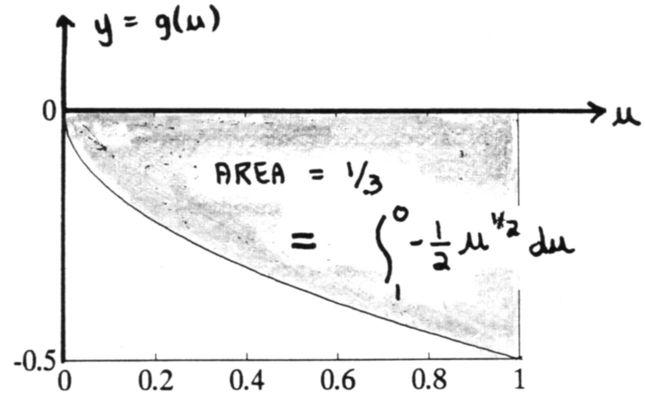
KEY OBSERVATIONS for using Approach #2

- Decide upon an appropriate substitution, just as you do with indefinite integral problems.
- Write the substitution directly under the definite integral, as usual.
- Directly under the substitution, calculate the limits of integration for the *new* definite integral (in the variable u). Remember: *don't change the limits of integration UNTIL you've rewritten the integral in terms of the new variable!*
- With this method, you never need to transform the antiderivative back to a function in the original variable.

Below is a sketch illustrating what is happening, from a graphical point of view, in this process.



Graph of $f(x) = x\sqrt{1-x^2}$ on $[0,1]$



Graph of $g(u) = -\frac{1}{2}u^{1/2}$ on $[0,1]$;

area beneath u-axis introduces a "$-$" sign;

integrating 'backwards' introduces another "$-$" sign!

variation on approach #2;

don't actually calculate the new limits, just note that they are different

There is a variation on the second approach that is often useful. Instead of *actually calculating* the new limits of integration, just make the reader aware that the limits have changed in the transformed problem. That is, when an 'old' limit of integration is 'a', the 'new' limit of integration is denoted by '$u(a)$' (the function u, evaluated at a). The technique is illustrated below:

Solution #3:

$$
\begin{aligned}
\int_0^1 x\sqrt{1-x^2}\,dx &= -\frac{1}{2}\int_0^1 -2x\sqrt{1-x^2}\,dx \\
&= -\frac{1}{2}\int_{u(0)}^{u(1)} u^{1/2}\,du \\
&= -\frac{1}{2}\frac{2}{3}u^{3/2}\Big|_{u(0)}^{u(1)} \\
&= -\frac{1}{3}(1-x^2)^{3/2}\Big|_0^1 \\
&= -\frac{1}{3}(0-1) = \frac{1}{3}
\end{aligned}
$$

$$\boxed{\begin{aligned} u &= 1-x^2 \\ du &= -2x\,dx \end{aligned}}$$

NOTE that the new limits are different but don't actually calculate them

CHANGE back to an antiderivative in x and use original limits

This technique is useful if the limits of integration for the transformed problem would be particularly messy, or difficult to compute.

EXERCISE 1

♣ Find $\int_0^1 x(3x^2-1)^5\,dx$. Write down your solution in three different ways. Be sure to write complete and correct mathematical sentences.

lurking in the background

The theoretical justification for this section lies in the following *change of variables formula*:

Change of Variables Formula

Let f and g' be continuous. Then:

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Observe that this formula states exactly what we've been doing in this section: letting $u = g(x)$, one obtains $du = g'(x)\,dx$; when $x = a$, $u = g(a)$ and when $x = b$, $u = g(b)$.

$$\int_a^b f(\overbrace{g(x)}^{u})\,\overbrace{g'(x)\,dx}^{du} = \int_{g(a)}^{g(b)} f(u)\,du$$

★★

existence of antiderivatives

If a function f is continuous on $[a,b]$, then the function F defined by

$$F(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$, differentiable on (a,b), and $F'(x) = f(x)$ for all $x \in (a,b)$.

Thus, every continuous function has an antiderivative. This fact is needed in (the first line of) the following proof.

PROOF
of the Change of Variables Formula

Proof. Let F be any antiderivative for f, so $F' = f$. Then, by the Chain Rule,

$$\frac{d}{dx}F(g(x)) = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$$

so that $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$. Thus:

$$\begin{aligned}\int_a^b f(g(x))\, g'(x)\, dx &= F(g(x))\big|_a^b \\ &= F(g(b)) - F(g(a))\end{aligned}$$

Also:

$$\begin{aligned}\int_{g(a)}^{g(b)} f(u)\, du &= F(u)\big|_{g(a)}^{g(b)} \\ &= F(g(b)) - F(g(a))\end{aligned}$$

Compare! ∎

using integration by parts with definite integrals

When using the integration by parts formula with definite integrals, one again has to be careful how things are written down.

As usual, one option is to first solve the corresponding indefinite integral problem, and use any antiderivative to evaluate the definite integral. However, it is more compact to evaluate the definite integral directly, as illustrated in the next example.

EXAMPLE
using parts with a definite integral

Problem: Find $\int_1^2 \ln x\, dx$.

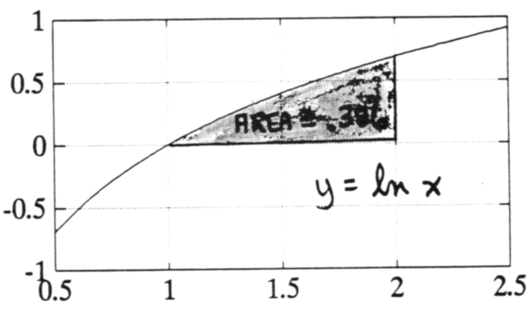
Solution:

$$\begin{aligned}\int_1^2 \ln x\, dx &= x \ln x\big|_1^2 - \int_1^2 x \cdot \frac{1}{x}\, dx \\ &= (2\ln 2 - 1\ln 1) - \Big[x\big|_1^2\Big] \\ &= 2\ln 2 - [2-1] \\ &= 2\ln 2 - 1 \approx 0.386\end{aligned}$$

$$\boxed{\begin{array}{ll} u = \ln x & dv = dx \\ du = \frac{1}{x}dx & v = x \end{array}}$$

Thus, the area under the graph of $y = \ln x$ on $[1, 2]$ is approximately 0.386 .

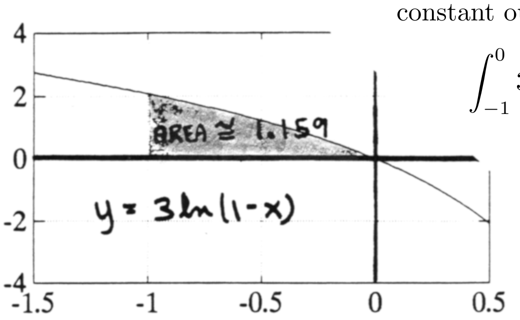
Note that it was necessary to evaluate *each part* of the antiderivative from 1 to 2 . In both cases, the symbol '$\big|_1^2$' is read as '*evaluated from* 1 *to* 2'.



EXAMPLE

Problem: find $\int_{-1}^{0} 3\ln(1-x)\,dx$.

Solution: It is usually easiest to use the linearity of the integral to factor the constant out first:



$$
\begin{aligned}
\int_{-1}^{0} 3\ln(1-x)\,dx &= 3\left[\int_{-1}^{0} \ln(1-x)\,dx\right] \\
&= 3\left[(x-1)\ln(1-x)\Big|_{-1}^{0} - \int_{-1}^{0} (x-1)\frac{1}{x-1}\,dx\right] \\
&= 3\left[(0+2\ln 2) - \int_{-1}^{0} (1)\,dx\right] \\
&= 3\left[2\ln 2 - x\Big|_{-1}^{0}\right] \\
&= 3[2\ln 2 - (0+1)] \\
&= 6\ln 2 - 3 \approx 1.159
\end{aligned}
$$

$u = \ln(1-x)$ $dv = dx$

$du = \frac{1}{1-x}(-1)\,dx$ $v = x-1$

$= \frac{1}{x-1}\,dx$

Observe how v was chosen to be $x-1$, instead of simply x, to simplify the integral $\int v\,du$.

EXERCISE 2

♣ Find $\int_0^1 xe^x\,dx$, by using parts. Do not solve the corresponding indefinite integral problem first; work directly with the definite integral.

QUICK QUIZ
sample questions

1. Find $\int_0^{\frac{1}{2}} (2x-1)^3\,dx$ by first solving the companion indefinite integral problem.
2. Find $\int_0^{\frac{1}{2}} (2x-1)^3\,dx$ by transforming it into a definite integral in the variable u, with correct limits of integration.
3. Solve $\int_1^e \ln x\,dx$ directly. That is, do NOT first solve the companion indefinite integral problem.

KEYWORDS
for this section

Various approaches to using the substitution technique in the context of definite integrals, the Change of Variables formula, using parts with definite integrals.

END-OF-SECTION EXERCISES

Evaluate the following definite integrals. Use any correct solution technique. Be sure to write complete mathematical sentences. Approximate answers to three decimal places.

1. $\displaystyle\int_{-1}^{1} x\sqrt{1+x^2}\,dx$
2. $\displaystyle\int_0^3 \frac{2}{3x+4}\,dx$
3. $\displaystyle\int_1^2 \frac{1}{(5-t)^3}\,dt$
4. $\displaystyle\int_1^3 \ln 3x\,dx$
5. $\displaystyle\int_2^3 5\ln(x-1)\,dx$