7.3 The Definite Integral as the Limit of Riemann Sums

Introduction

This section presents the actual *definition* of the definite integral. As previously noted, one is often able to bypass this definition, due to the Fundamental Theorem of Integral Calculus. However, *it is still extremely important that you see this definition*, for three reasons:

- The definition provides the motivation for the notation
$$\int_a^b f(x)\,dx$$
that is used in connection with the definite integral.
- The definition provides the *intuition* that mathematicians use to help them develop many useful formulas involving the definite integral; e.g., finding the area between two curves and finding volumes of revolution. These formulas are presented later on in this chapter.
- The definition provides the justification for numerical methods used to approximate $\int_a^b f(x)\,dx$, when one is unable to obtain an antiderivative of f.

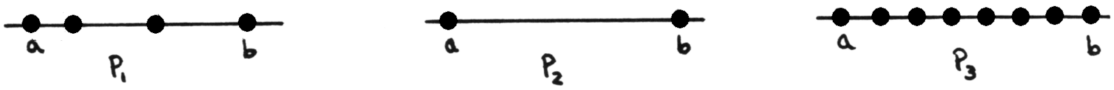
EXERCISE 1	♣ What are the three reasons for which it is important that you see the *definition* of the definite integral?

partition of an interval $[a,b]$

We begin with some definitions.

A *partition* of the interval $[a,b]$ is a finite collection (set) of points from $[a,b]$ that includes the endpoints a and b.

Some partitions of $[a,b]$ are shown below:



By convention, when one writes a partition

$$P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$$

of $[a,b]$, it is assumed that:

- $x_0 = a$; that is, the first point in the partition is the left-hand endpoint a
- $x_n = b$; that is, the last point in the partition is the right-hand endpoint b
- The points are listed in *increasing* order, so that:

$$x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n$$

Observe that a partition of $[a,b]$ naturally breaks the interval $[a,b]$ into *non-overlapping subintervals* whose union is the entire interval $[a,b]$:

$$[\overbrace{x_0}^{=a}, x_1) \cup [x_1, x_2) \cup \cdots \cup [x_{n-2}, x_{n-1}) \cup [x_{n-1}, \overbrace{x_n}^{=b}]$$

EXERCISE 2

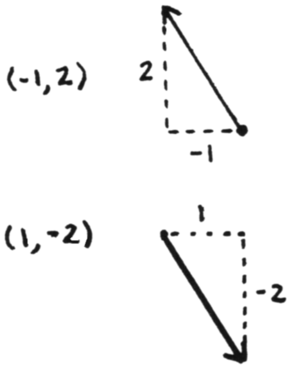
♣ 1. How many points are in the partition $P = \{1\,,\,2\,,\,2.5\,,\,3\}$ of $[1,3]$? Show these points on a number line. Into how many subintervals is $[1,3]$ divided by this partition?

♣ 2. How many points are in the partition $P = \{x_0, x_1, \dots, x_n\}$ of an interval $[a,b]$? Into how many subintervals is $[a,b]$ divided by this partition?

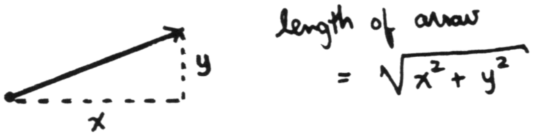
norm

A *norm* is a tool used in mathematics to measure the *size* of objects.

For example, the absolute value $|\cdot|$ measures the size of real numbers; the function that maps a real number x to its 'size' $|x|$ is a *norm* on $\mathbb{R}$.



As a second example, a natural way to 'measure the size' of a pair of real numbers (x,y) is to first look at the arrow (vector) representing (x,y), and then measure its length;



the function that maps a pair (x,y) of real numbers to its 'size' $\sqrt{x^2+y^2}$ is a *norm* on the set of all ordered pairs.

measuring the 'size' of a partition

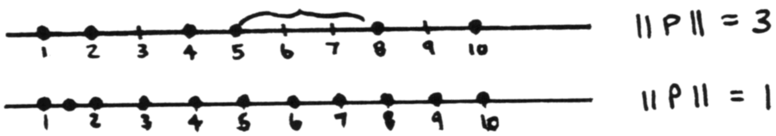
We need a way of measuring the *size* of a partition of $[a,b]$. We want to say that the partition is 'small' if the lengths of *all* the subintervals are small. Observe that if the length of the *longest* subinterval is small, then the lengths of *all* the subintervals must be small. This motivates the next definition.

norm of a partition; $\|P\|$

Define $\|P\|$ (read as the '*norm of the partition P*') to be the length of the *longest* subinterval in the partition P.

For example, if P is the partition $\{1,2,4,5,8,10\}$ of $[1,10]$, then $\|P\| = 3$, since the length of the longest subinterval is 3.

Also, if $P = \{1\,,\,1.5\,,\,2\,,\,3\,,\,4\,,\,5\,,\,6\,,\,7\,,\,8\,,\,9\,,\,10\}$, then $\|P\| = 1$, since the length of the longest subinterval is 1.



The closer $\|P\|$ is to zero, the smaller the subintervals, and hence the more points there are in P.

EXERCISE 3

♣ 1. Give a partition of $[0,1]$ that has norm $\frac{1}{2}$. How many points are in this partition?

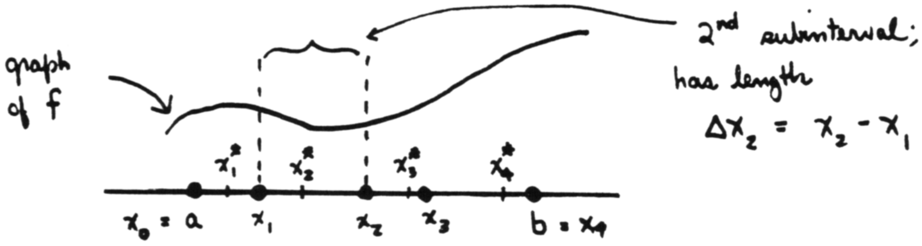
♣ 2. Give a different partition of $[0,1]$ that has norm $\frac{1}{2}$. How many points are in this partition?

♣ 3. What are the *fewest* number of points that you must have in a partition of $[0,1]$, in order for it to have norm $\frac{1}{2}$?

Riemann Sum for f;

x_i^* is our choice from the i^{th} subinterval, which has length Δx_i

Let f be continuous on $[a, b]$, and let $P = \{x_0, \dots, x_n\}$ be any partition of the interval $[a, b]$, as illustrated below.



In each of the n subintervals, choose *any point*; let x_i^* denote the choice from the i^{th} subinterval.

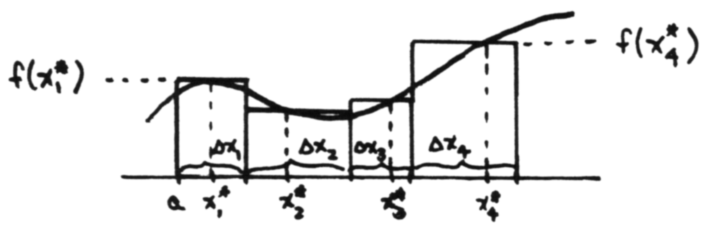
Also, let $\Delta x_i := x_i - x_{i-1}$ denote the length of the i^{th} subinterval.

Then, the sum

$$R(P) := f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

is called a *Riemann sum for f*, corresponding to the partition P. ('Riemann' is pronounced REE-mon.)

Observe that if f is nonnegative, then the sum $R(P)$ represents the sum of the areas of the rectangles shown below, which approximates the area under the graph of f on $[a, b]$.



EXERCISE 4

Consider the partition $P = \{0, 1, 2, 3, 4\}$ of $[0, 4]$. Let $f(x) = x^2$.

♣ 1. Choose the midpoint from each subinterval of P. That is, choose:

$$x_1^* = 0.5\,,\ x_2^* = 1.5\,,\ x_3^* = 2.5\,,\ \ x_4^* = 3.5$$

Make a sketch that shows the graph of f, the partition P, and the choices x_i^*.

♣ 2. On each subinterval, draw a rectangle with height $f(x_i^*)$.

♣ 3. Sum the areas of these rectangles. That is, find the Riemann sum for f corresponding to the choices x_i^*.

♣ 4. What is the *actual* area under the graph under f on $[0, 4]$?

EXERCISE 5 ♣ Repeat the previous exercise, except this time with the partition

$$\{0\,,\,0.5\,,\,1\,,\,1.5\,,\,2\,,\,2.5\,,\,3\,,\,3.5\,,\,4\}$$

of $[0,4]$. Again choose the x_i^* to be the midpoints of each subinterval.

This time, what is the Riemann sum for f corresponding to the partition P and choices x_i^*?

obtain the definite integral by letting $\|P\| \rightarrow 0$

Under the hypothesis that f is continuous on $[a,b]$, it can be proven that as one chooses partitions with smaller and smaller norms, the corresponding Riemann sums approach a unique number.

We *define* this unique number to be the *definite integral of f on* $[a,b]$, denoted by $\int_a^b f(x)\,dx$.

more precisely

More precisely, as $\|P\| \rightarrow 0$, $R(P) \rightarrow \int_a^b f(x)\,dx$.

That is, we can get the numbers $R(P)$ as close to $\int_a^b f(x)\,dx$ as desired, merely by choosing a partition P of $[a,b]$ with norm sufficiently close to 0.

In other words, for every $\epsilon > 0$, there exists $\delta > 0$, such that if a partition P is chosen with $\|P\| < \delta$, then:

$$\left| R(P) - \int_a^b f(x)\,dx \right| < \epsilon$$

Rephrasing yet one more time, we can get the Riemann sum $R(P)$ as close to the number $\int_a^b f(x)\,dx$ as desired, by choosing a partition P of $[a,b]$ that has sufficiently small subintervals.

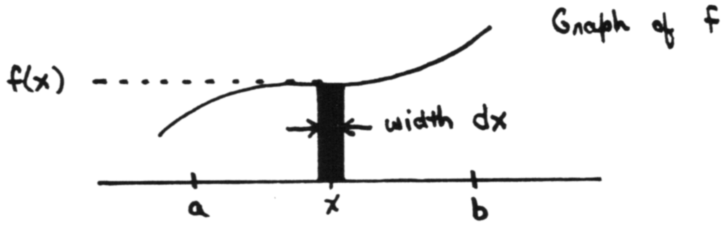
It is clear from the definition of $\int_a^b f(x)\,dx$ that this integral gives information about the *area* trapped between the graph of f and the x-axis.

If f is positive on $[a,b]$, then any Riemann sum $R(P)$ is also positive, and approximates the area under the graph of f on $[a,b]$.

If f is negative on $[a,b]$, then any Riemann sum $R(P)$ is also negative. (♣ Why?) The magnitude of the negative number $R(P)$ approximates the area trapped between the graph of f and the x-axis on $[a,b]$.

motivation for the notation $\int_a^b f(x)\,dx$; $f(x)\,dx$ is the (signed) area of a rectangle, with width dx, and height $f(x)$

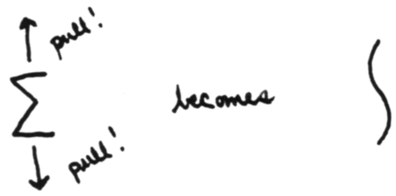
The definition of the definite integral of f on $[a,b]$ provides the motivation for the notation $\int_a^b f(x)\,dx$ used, as follows:



Think of dx as an *infinitesimally small piece of the x-axis.* At a point x between a and b, construct a rectangle of width dx and height $f(x)$. Then (using calculus!) 'sum' these rectangles as x varies from a to b .

$\sum$ becomes $\int$

The integral sign $\int$ is, therefore, a kind of *super sum*; indeed, one can think of obtaining it from the summation sign $\sum$ used for finite sums by stretching it out!



integration is an (infinite) summation process

That is, *integration is really an (infinite) summation process.*

If seeing the notation $\int_a^b f(x)\,dx$ conjures an image of a limit of Riemann sums, then it is a successful notation.

QUICK QUIZ
sample questions

1. What is a *partition* of an interval $[a, b]$?
2. Give two different partitions of $[1, 3]$ that have norm $1/2$.
3. Let $f(x) = x^2$, and take the partition $\{0, 1, 2, 3\}$ of the interval $[0, 3]$. Is there a unique Riemann sum for f corresponding to this partition? Comment.
4. What picture might you think of when you see the notation $\int_a^b f(x)\,dx$?

KEYWORDS
for this section

Three reasons for seeing the definition of the definite integral, partition of an interval, norm, norm of a partition, Riemann sum for f, obtain the definite integral by letting $\|P\| \to 0$, motivation for the notation $\int_a^b f(x)\,dx$, integration is an (infinite) summation process.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN).

♣ For any *sentence*, state whether it is TRUE (T), FALSE (F), or CONDITIONAL (C).

1. $\int x^2\,dx$
2. $\int_0^1 x^2\,dx$
3. $\int_0^1 x^2\,dx = \frac{1}{3}$
4. The integral $\int_a^b f(x)\,dx$ gives the magnitude of the area bounded between the graph of f and the x-axis on $[a, b]$.
5. If $a < b$, then the integral $\int_a^b e^x$ gives the magnitude of the area bounded between the graph of $y = e^x$ and the x-axis on $[a, b]$.
6. If P is a partition of $[a, b]$, then a Riemann sum $R(P)$ corresponding to f is an approximation to $\int_a^b f(x)\,dx$.
7. If g is twice differentiable on the interval $[a, b]$, then $\int_a^b g'(x)\,dx = g(b) - g(a)$.
8. If $a < b$ and f is continuous on $[a, b]$, then $\int_a^b |f(x)|\,dx \ge 0$.
9. If $a < b$ and f is continuous on $[a, b]$, then $\int_a^b (-|f(x)|)\,dx \le 0$.
10. For all real numbers a and b, $\int_a^b x^2\,dx = \int_a^b t^2\,dt$.