6.5 More on Substitution

integration is more difficult than differentiation

Integration is more difficult than differentiation. To *differentiate* a function, one only needs to recognize the *form* of the function—a product, a quotient, a composite function—and then use the appropriate differentiation tool. There is a much bigger 'bag of tricks' associated with integration. For the most part, people who are good at integrating are people who have had *lots of experience* integrating. Over time, with lots of practice, you will learn to recognize different types of integration problems, and apply appropriate tools.

more advanced substitution techniques

The substitution technique discussed in the previous section is the 'basic model' of substitution. In this section, more advanced substitution techniques are investigated.

a slight twist on the 'basic model' of substitution

Consider the integration problem:

$$\int x(x+1)^{10}\,dx$$

From a theoretical viewpoint, since $x(x+1)^{10}$ is just a polynomial, the problem is easy. From a computational viewpoint, one certainly doesn't want to multiply out $(x+1)^{10}$. And, the 'basic model' of substitution doesn't seem to work at first glance: one could try letting $u = x+1$, but there's an extra 'x' in the integrand, that cannot be pulled out of the integral.

Study the next example, to see how this 'problem' is overcome.

EXAMPLE

Problem: Solve $\int x(x+1)^{10}\,dx$.

Solution: Define $u := x+1$. Then, $du = dx$, and (writing x in terms of u), $x = u-1$. Transforming the integral in x to an integral in u yields:

$u = x+1$; $x = u-1$
$du = dx$

$$\begin{aligned}\int x(x+1)^{10}\,dx &= \int (u-1)u^{10}\,du\\ &= \int u^{11} - u^{10}\,du\\ &= \frac{u^{12}}{12} - \frac{u^{11}}{11} + C\\ &= \frac{(x+1)^{12}}{12} - \frac{(x+1)^{11}}{11} + C\end{aligned}$$

What made this work? Firstly, it was possible to rewrite the entire integrand in terms of u. Secondly, the resulting function of u was easier to integrate than the original function of x.

'role reversal'

Note that the substitution $u = x+1$ in the previous example transformed

$$\int x(x+1)^{10}\,dx \quad \text{to} \quad \int (u-1)u^{10}\,du\ ;$$

in the first integral, the *sum* is raised to the tenth power, and in the second integral, the *singleton* is raised to the tenth power. Hence, the substitution provided a sort of 'role reversal'. The next few examples illustrate the use of substitution for this type of 'role reversal'.

EXAMPLE
'role reversal'

Problem: Find $\int \frac{x}{x+1}\,dx$.

Solution: The problem is of the form $\int \frac{\text{singleton}}{\text{sum}}\,dx$. If the problem were instead of the form $\int \frac{\text{sum}}{\text{singleton}}\,dx$, then it would be easy, since, for example, $\frac{x+1}{x} = 1 + \frac{1}{x}$. Thus, the denominator is 'transformed to a singleton' by defining $u := x + 1$:

$$\begin{aligned} \int \frac{x}{x+1}\,dx &= \int \frac{u-1}{u}\,du \\ &= \int (1 - \frac{1}{u})\,du \\ &= u - \ln|u| + C \\ &= (1+x) - \ln|1+x| + C \\ &= x - \ln|1+x| + K \end{aligned}$$

$u = x+1\,;\ x = u-1$
$du = dx$

In the last step, the constant 1 was absorbed into the constant of integration, to obtain a simpler answer.

Check: $\frac{d}{dx}(x - \ln|1+x|) = 1 - \frac{1}{1+x} = \frac{1+x-1}{1+x} = \frac{x}{1+x}$

alternate solution; long division

Here's an alternate solution to the integration problem $\int \frac{x}{1+x}\,dx$.

Alternate Solution: First, do a long division. Remember that when you divide by a polynomial, you want to write the divisor so that the powers of x decrease as you go from left to right:

$$\begin{array}{r|l} & \ \ 1 \\ \hline x+1 & \ \ x \\ & -(x+1) \\ \hline & \ \ -1 \end{array}$$

Thus, $\frac{x}{x+1} = 1 - \frac{1}{x+1}$. Then:

$$\begin{aligned} \int \frac{x}{x+1}\,dx &= \int (1 - \frac{1}{x+1})\,dx \\ &= x - \ln|x+1| + C \end{aligned}$$

EXERCISE 1 ♣ Find $\int \frac{3t}{t-1}\,dt$ in two ways. First, use the 'role reversal' substitution technique. Second, use long division.

EXAMPLE

Problem: Find all the antiderivatives of $\dfrac{3t}{2t+1}$.

Solution: In problems such as this, it is often easier to keep track of things if 'dt' is solved for in terms of 'du':

$$
\begin{aligned}
\int \frac{3t}{2t+1}\,dt &= 3\int \frac{t}{2t+1}\,dt \\
&= 3\int \frac{\frac{u-1}{2}}{u}\,\frac{du}{2} \\
&= \frac{3}{4}\int \frac{u-1}{u}\,du \\
&= \frac{3}{4}\int 1-\frac{1}{u}\,du \\
&= \frac{3}{4}(u-\ln|u|)+C \\
&= \frac{3}{4}(2t+1-\ln|2t+1|)+C \\
&= \frac{3}{4}(2t-\ln|2t+1|)+K
\end{aligned}
$$

$u = 2t+1$;
$t = \dfrac{u-1}{2}$

$du = 2dt$;
$dt = \dfrac{du}{2}$

The technique worked, because it was possible to rewrite the integrand entirely in terms of u, AND the resulting function of u was easier to integrate than the initial function of x.

♣ What was done in the last step of the previous integration?

EXERCISE 2

♣ Evaluate $\int \frac{2t}{3t-1}\,dt$ in two ways. First, use the 'role reversal' substitution technique. Second, use long division.

EXAMPLE

Problem: Find $\displaystyle\int \frac{3x}{(2x-1)^5}\,dx$.

Solution:

$$
\begin{aligned}
\int \frac{3x}{(2x-1)^5}\,dx &= 3\int \frac{(u+1)/2}{u^5}\,\frac{du}{2} \\
&= \frac{3}{4}\int \frac{u+1}{u^5}\,du \\
&= \frac{3}{4}\int (u^{-4}+u^{-5})\,du \\
&= \frac{3}{4}\left(\frac{u^{-3}}{-3}+\frac{u^{-4}}{-4}\right)+C \\
&= \frac{3}{4}\left(-\frac{1}{3(2x-1)^3}-\frac{1}{4(2x-1)^4}\right)+C
\end{aligned}
$$

$u = 2x-1$;
$x = \dfrac{u+1}{2}$

$du = 2dx$;
$dx = \dfrac{du}{2}$

EXAMPLE

Problem: Find $\int \frac{x}{2\sqrt{3x-1}}\,dx$.

Solution:

$$\begin{aligned}\int \frac{x}{2\sqrt{3x-1}}\,dx &= \frac{1}{2}\int \frac{u+1}{3}\cdot\frac{1}{\sqrt{u}}\,\frac{du}{3}\\ &= \frac{1}{18}\int \frac{u+1}{u^{1/2}}\,du\\ &= \frac{1}{18}\int (u^{1/2}+u^{-1/2})\,du\\ &= \frac{1}{18}(\frac{2}{3}u^{3/2}+2u^{1/2})+C\\ &= \frac{1}{18}\big(\frac{2}{3}(3x-1)^{3/2}+2(3x-1)^{1/2}\big)+C\\ &= \frac{1}{27}\sqrt{(3x-1)^3}+\frac{1}{9}\sqrt{3x-1}+C\end{aligned}$$

$u = 3x-1;$

$x = \frac{u+1}{3}$

$du = 3dx;$

$dx = \frac{du}{3}$

EXERCISE 3 Solve the following integration problems. Use any appropriate techniques.

♣ 1. $\int t(t+1)^7\,dt$

♣ 2. $\int \frac{5x}{\sqrt{(3-2x)^3}}\,dx$

♣ 3. $\int u\sqrt{u^2+1}\,du$

rationalizing substitutions

Remember that to 'rationalize' means to 'get rid of the radical'. Sometimes, an appropriate substitution can be used to get rid of a radical, and transform a difficult problem into a more manageable one. The technique is illustrated in the next example.

EXAMPLE

a rationalizing substitution

Problem: Find $\int \frac{1}{1+\sqrt{x}}\,dx$.

Solution: To *rationalize the integrand*, let $u=\sqrt{x}$, so that $u^2=x$. Remember that u is a function of x, and differentiate both sides of $u^2=x$ with respect to x, getting:

$$2u\frac{du}{dx}=1$$

Thus,

$$2u\,du = dx\ .$$

Now, transforming to an integral in u yields:

$$\begin{aligned}\int \frac{1}{1+\sqrt{x}}\,dx &= \int \frac{1}{1+u}\,(2u\,du)\\ &= 2\int \frac{u}{1+u}\,du\end{aligned}$$

$u = \sqrt{x};$

$u^2 = x$

$2u\frac{du}{dx} = 1;$

$2u\,du = dx$

At this point, the previous *reversal of roles* procedure can be used:

$w = 1 + u;$
$u = w - 1$
$dw = du$

$$\begin{aligned}
2\int \frac{u}{1+u}\,du &= 2\int \frac{w-1}{w}\,dw \\
&= 2\int 1 - \frac{1}{w}\,dw \\
&= 2(w - \ln|w|) + C \\
&= 2\big((1+u) - \ln|1+u|\big) + C \\
&= 2u - 2\ln|1+u| + K \\
&= 2\sqrt{x} - 2\ln|1+\sqrt{x}| + K
\end{aligned}$$

Remember that since we started with an integration problem involving x, it was necessary to end up with the antiderivatives in terms of x.

EXERCISE 4

♣ 1. Re-do the previous problem, without looking at the text.

♣ 2. Check that: $\dfrac{d}{dx}(2\sqrt{x} - 2\ln|1+\sqrt{x}|) = \dfrac{1}{1+\sqrt{x}}$

EXERCISE 5

♣ Solve the integral $\int \frac{x}{\sqrt{x-1}}\,dx$ in two ways. First, let $u = x-1$ and make a 'role reversal'. Second, let $u = \sqrt{x-1}$, so that $u^2 = x - 1$, and make a rationalizing substitution. Compare your answers. Which way do you think was easier?

tables of integrals

In closing, it must be remarked that there are extensive tables of integrals available. One such compilation is:

Tables of Integrals and other Mathematical Data
Herbert Bristol Dwight, third edition
The MacMillan Company, New York, 1957

(This was my Dad's, so it is very special to me! There are obviously newer books available.)

To use such tables, one identifies the *form* of the integrand, finds a corresponding form in the table, and applies the formula.

For example, suppose one must integrate:

$$\int \frac{1}{x(1+3x^7)}\,dx$$

One finds the following entry in a table of integrals:

$$\int \frac{dx}{x(a+bx^m)} = \frac{1}{am}\log\left|\frac{x^m}{a+bx^m}\right|$$

Letting $a = 1$, $b = 3$, and $m = 7$, one applies the formula, getting:

$$\int \frac{1}{x(1+3x^7)}\,dx = \frac{1}{7}\log\left|\frac{x^7}{1+3x^7}\right|$$

♣ Check!

QUICK QUIZ
sample questions

1. Which is harder, in general, differentiation or integration?
2. Find all the antiderivatives of $\frac{x}{2+x}$. Use any appropriate technique.
3. What tools are available to help with integration?

KEYWORDS
for this section

Reversal of roles substitution technique, a rationalizing substitution, tables of integrals.

END-OF-SECTION EXERCISES

♣ The purpose of these exercises is to provide you with additional practice using *all* the antidifferentiation techniques discussed thus far in this chapter. Be sure to write complete mathematical sentences.

1. $\displaystyle\int \frac{e^{2x}+1}{5}\,dx$
2. $\displaystyle\int xe^{(3x^2-1)}\,dx$
3. $\displaystyle\int \frac{t}{\sqrt[3]{4t^2-1}}\,dt$
4. $\displaystyle\int \frac{x}{2x-1}\,dx$
5. $\displaystyle\int x(x+1)^3(x-1)^3\,dx$
6. $\displaystyle\int \frac{2t-1}{t}\,dt$
7. $\displaystyle\int \frac{(\ln x)^3}{3x}\,dx$