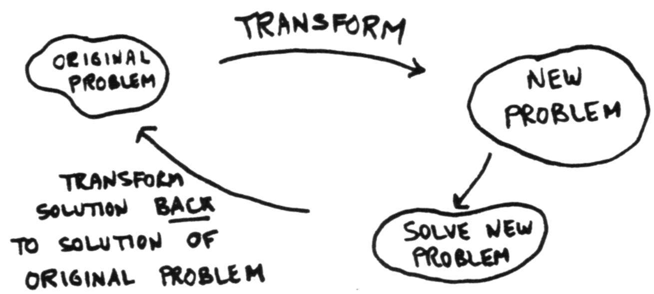
6.4 The Substitution Technique for Integration

a recurrent theme in mathematics; transforming a difficult problem into an easier one

A recurrent theme in mathematics is that of *transforming* a problem that is difficult to solve into one that is easier to solve.

This idea has already been used extensively: in the process of solving an equation, one *transforms* the original equation into an equivalent one (that is, one with the same solution set) that is easier to work with.

In this section, a method is studied by which it is often possible to transform a difficult integration problem into one that is much easier. The *transformed* problem is then solved, and the solution used to obtain the solution of the *original* problem. The technique is referred to as *substitution.*



EXAMPLE

the substitution technique for integration

Here's an example that illustrates the technique. Suppose one wants to find:

$$\int (3-4x^2)^{100}(-8x)\,dx$$

Theoretically at least, this problem *is* solvable with the tools currently available: one need 'only' multiply out $(3-4x^2)^{100}$, multiply this by $-8x$, and then integrate the resulting polynomial term-by-term. Practically speaking,

there must be a better way,

and there is.

do some renaming

Let's do some 'renaming'. Define a new variable u by $u := 3 - 4x^2$, and differentiate to see that $\frac{du}{dx} = -8x$. There just happens to be a $-8x$ in the integrand. So, the integral can be rewritten in terms of u:

$$\int (\overbrace{3 - 4x^2}^{u})^{100} \overbrace{(-8x)}^{\frac{du}{dx}} \, dx \;=\; \int u^{100} \frac{du}{dx} \, dx$$

Motivated by 'cancelling the dx's', one might conjecture that an equivalent problem is

$$\int u^{100} \, du \ ,$$

which is a problem that *can* be solved easily: $\int u^{100} \, du = \frac{u^{101}}{101} + C$

♣ What is a 'conjecture'?

Indeed, $\frac{u^{101}}{101} + C$ *is* the solution of $\int u^{100} \frac{du}{dx} \, dx$, since by the extended power rule for differentiation:

$$\frac{d}{dx} \frac{u^{101}}{101} = \frac{1}{101}(101u^{101-1})\frac{du}{dx} = u^{100}\frac{du}{dx}$$

(Remember that u is a function of x, and differentiate accordingly.) Next, transform the solution $\frac{u^{101}}{101} + C$ back to the variable x. Since $u = 3 - 4x^2$, the solution to the original problem is:

$$\int (3 - 4x^2)^{100}(-8x) \, dx = \frac{(3 - 4x^2)^{101}}{101} + C$$

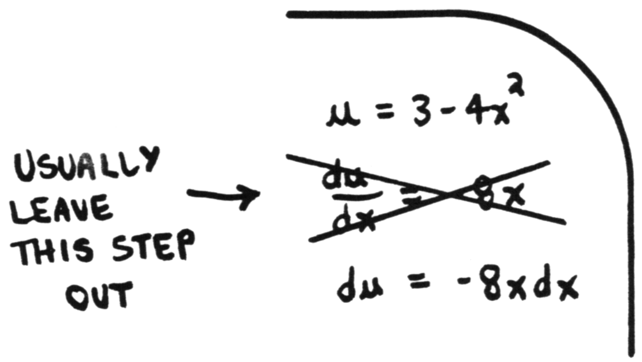
EXERCISE 1 ♣ Check, by differentiating, that:

$$\int (3 - 4x^2)^{100}(-8x) \, dx = \frac{(3 - 4x^2)^{101}}{101} + C$$

simplified notation for the previous problem

Henceforward, here's how the previous problem will be written down:

$$\int (\overbrace{3 - 4x^2}^{u})^{100} \overbrace{(-8x) \, dx}^{du} = \int u^{100} \, du$$



$$= \frac{u^{101}}{101} + C$$

$$= \frac{(3 - 4x^2)^{101}}{101} + C$$

Observe several important features of this solution:

- Write the substitution ($u = 3 - 4x^2$, in this case) *directly under* the integration problem.
- When u is a function of x, du is found by first differentiating u with respect to x
$$\frac{du}{dx} = -8x$$
and then 'multiplying' both sides by dx to obtain du. The justification for this procedure was motivated by the first example.
Usually, one doesn't bother to write down the intermediate step $\frac{du}{dx} = -8x$.
- Line up the equal signs as you are solving the problem. This form makes it easy to see the *original integration problem* and the *solution* at a glance.
- Once the solution in terms of the new variable u is obtained, rewrite this solution in terms of the original variable, x.

EXERCISE 2

♣ Supply a reason for each step:

$$\int (\overbrace{3 - 4x^2}^{u})^{100} \overbrace{(-8x)\,dx}^{du} = \int u^{100}\,du$$

$$= \frac{u^{101}}{101} + C$$

$$= \frac{(3 - 4x^2)^{101}}{101} + C$$

$$u = 3 - 4x^2$$
$$du = -8x\,dx$$

Don't mix variables!

Don't ever 'mix' variables when writing down your solution, like in:

$$\int (3 - 4x^2)^{100} x\,dx = \int \underbrace{\overbrace{u^{100}x}^{\text{BAD!}}}_{u \text{ and } x \text{ mixed}} \; dx = \cdots$$

Get everything ready to change to the new variable, and then do it—all at once.

choosing a 'u that works'

Strategy: choose something for u such that $\frac{du}{dx}$ also appears in the integrand

Not all problems are solvable by substitution, but many are. If you are faced with a difficult integration problem, the technique of substitution should always be tried. The challenge is, of course, to find a choice for u that 'works'. Here's the general strategy:

- Choose something for u so that its derivative $\frac{du}{dx}$ appears as a factor in the integrand (possibly off by a constant).

Often, as examples will illustrate, u is something that is raised to a power, or under a radical.

In the previous example, u was chosen to be $3-4x^2$ because it was noted that the derivative, $-8x$, was also a factor in the integrand. Actually, it is only critical that the *variable part* of the derivative appear in the integrand; linearity of the integral can be used to take care of *constants*, as the next example illustrates.

EXAMPLE
introducing a constant; multiply by 1 in an appropriate form

Problem: Evaluate $\int (3-4x^2)^{100}\,x\,dx$.

Solution: Note the similarity to the previous example. The only difference is that this time the '-8' is missing.

The substitution $u = 3-4x^2$ is still a good choice, since $\frac{d}{dx}(3-4x^2) = -8x$, and the *variable* part of this derivative, x, appears as a factor in the integrand.

To transform the problem into an integral in u, it is necessary to bring a -8 into the picture, *without changing the problem.* This can be accomplished by the usual technique of *multiplying by* 1 *in an appropriate form*:

$$\int (3-4x^2)^{100}x\,dx = \int (3-4x^2)^{100}\Big(\frac{-8}{-8}\Big)x\,dx \qquad \text{(multiply by 1 in form } \frac{-8}{-8}\text{)}$$

$u = 3-4x^2$
$du = -8x\,dx$

$$= \frac{1}{-8}\int \overbrace{(3-4x^2)}^{u}{}^{100}\,\overbrace{(-8x)\,dx}^{du} \qquad \text{(linearity of integral)}$$

$$= -\frac{1}{8}\int u^{100}\,du \qquad \text{(transform to } u\text{)}$$

$$= -\frac{1}{8}\cdot\frac{u^{101}}{101} + C \qquad \text{(solve problem in } u\text{)}$$

$$= -\frac{1}{8}\cdot\frac{(3-4x^2)^{101}}{101} + C \qquad \text{(rewrite in } x\text{)}$$

Since *constants* can be 'slid out' of the integral, we were able to 'get rid of' the undesired '$\frac{1}{-8}$' in the integrand. Only the -8 was left in the integrand, since this was needed as part of *du*.

EXERCISE 3

♣ 1. Check, by differentiating, that:

$$\int (3-4x^2)^{100}x\,dx = -\frac{1}{8}\cdot\frac{(3-4x^2)^{101}}{101} + C$$

♣ 2. Where and how was the linearity of the integral used in arriving at this solution?

The technique of substitution is further illustrated with a number of examples. Pay particular attention to the *complete mathematical sentences* in each of these examples.

EXAMPLE
evaluate an integral

Problem: Evaluate $\int (t+10)^7\,dt$.

Solution:

$$\int \overbrace{(t+10)}^{u}{}^{7}\,\overbrace{dt}^{du} = \int u^7\,du$$

$u = t+10$
$du = dt$

$$= \frac{u^8}{8} + C$$

$$= \frac{(t+10)^8}{8} + C$$

Check: $\dfrac{d}{dt}\dfrac{(t+10)^8}{8} = \dfrac{1}{8}\cdot 8(t+10)^7(1) = (t+10)^7$

EXAMPLE

find all the antiderivatives of a function

Problem: Find all the antiderivatives of $\dfrac{x^2}{\sqrt{x^3-1}}$.

Solution:

$$u = x^3 - 1$$
$$du = 3x^2\,dx$$

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x^3-1}}\,dx &= \frac{1}{3}\int \frac{3x^2}{\sqrt{x^3-1}}\,dx \\
&= \frac{1}{3}\int \frac{1}{\sqrt{u}}\,du \\
&= \frac{1}{3}\int u^{-1/2}\,du \\
&= \frac{1}{3}\cdot\frac{u^{1/2}}{1/2} + C \\
&= \frac{2}{3}\sqrt{x^3-1} + C
\end{aligned}$$

EXERCISE 4

♣ 1. Why was u chosen to be $x^3 - 1$ in the previous example?

♣ 2. Supply reasons for each step in the previous example. In particular, make sure you identify where the linearity of the integral was used.

♣ 3. Check the previous solution, by differentiating.

EXAMPLE

integrate

Problem: Integrate: $\displaystyle\int \frac{y+1}{(y^2+2y+1)^3}\,dy$

Solution:

$$u = y^2 + 2y + 1$$
$$du = (2y + 2)\,dy$$

$$\begin{aligned}
\int \frac{y+1}{(y^2+2y+1)^3}\,dy &= \int \frac{(\frac{1}{2})(2)(y+1)}{(y^2+2y+1)^3}\,dy \\
&= \frac{1}{2}\int \frac{2y+2}{(y^2+2y+1)^3}\,dy \\
&= \frac{1}{2}\int \frac{1}{u^3}\,du \\
&= \frac{1}{2}\int u^{-3}\,du \\
&= \frac{1}{2}\cdot\frac{u^{-2}}{-2} + C \\
&= -\frac{1}{4u^2} + C \\
&= -\frac{1}{4(y^2+2y+1)^2} + C
\end{aligned}$$

EXERCISE 5

♣ 1. Why was u chosen to be $y^2 + 2y + 1$ in the previous example?

♣ 2. Rewrite the previous example, using the dummy variable x instead of the dummy variable y. Do not look at the text while you are solving the problem.

♣ 3. Check the solution to the previous example, by differentiating.

EXAMPLE
two different approaches to the same problem

Problem: Find $\int e^{4+x}\,dx$ in two different ways.

Old way:

$$\begin{aligned}\int e^{4+x}\,dx &= \int e^4 e^x\,dx \\ &= e^4 \int e^x\,dx \\ &= e^4 \cdot e^x + C \\ &= e^{4+x} + C\end{aligned}$$

New way:

$$\begin{aligned}\int e^{4+x}\,dx &= \int e^u\,du \\ &= e^u + C \\ &= e^{4+x} + C\end{aligned}$$

$u = 4+x$
$du = dx$

Which was easier?

EXAMPLE
$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$

Problem: Find a formula for integrating e^{kx}, for any nonzero constant k.
Solution:

$$\begin{aligned}\int e^{kx}\,dx &= \frac{1}{k}\int k \cdot e^{kx}\,dx \\ &= \frac{1}{k}\int e^u\,du \\ &= \frac{1}{k}e^u + C \\ &= \frac{1}{k}e^{kx} + C\end{aligned}$$

$u = kx$
$du = k\,dx$

This is a nice formula to remember. Thus, for example:

$$\int 7e^{3x}\,dx = 7(\frac{1}{3})e^{3x} + C = \frac{7}{3}e^{3x} + C$$

EXAMPLE

Some people take a slightly different approach when solving problems like $\int e^{4+x}\,dx$ and $\int e^{3x}\,dx$, as illustrated below:

$$\begin{aligned}\int e^{4+x}\,dx &= \int du \\ &= u + C \\ &= e^{4+x} + C\end{aligned}$$

$u = e^{4+x}$
$du = e^{4+x}\,dx$

$$\begin{aligned}\int e^{3x}\,dx &= \frac{1}{3}\int 3e^{3x}\,dx \\ &= \frac{1}{3}\int du \\ &= \frac{1}{3}u + C \\ &= \frac{1}{3}e^{3x} + C\end{aligned}$$

$u = e^{3x}$
$du = 3e^{3x}\,dx$

Variety is the spice of life. Which way do you prefer?

EXAMPLE
finding a particular solution

Problem: Find a function f satisfying the following two conditions:

- the graph of f passes through the point $(0, 1)$
- $f'(x) = \dfrac{1}{3x+5}$

Solution: First, find ALL functions f that have derivative $\frac{1}{3x+5}$. That is, find all the antiderivatives of f':

$$u = 3x + 5$$
$$du = 3\,dx$$

$$\begin{aligned}
f(x) &= \int f'(x)\,dx \\
&= \int \frac{1}{3x+5}\,dx \\
&= \frac{1}{3}\int \frac{3}{3x+5}\,dx \\
&= \frac{1}{3}\int \frac{1}{u}\,du \\
&= \frac{1}{3}\ln|u| + C \\
&= \frac{1}{3}\ln|3x+5| + C
\end{aligned}$$

A problem like this was integrated earlier in the chapter, via a different technique. (See, for example, page 350.) Which technique do you prefer?

Check: Remember:

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

An application of the Chain Rule gives:

$$\frac{d}{dx}\ln|f(x)| = \frac{1}{f(x)} \cdot f'(x)$$

Then: $\frac{d}{dx}(\frac{1}{3}\ln|3x+5|) = \frac{1}{3} \cdot \frac{1}{3x+5} \cdot 3 = \frac{1}{3x+5}$

Second, choose the antiderivative that passes through the desired point:

$$\begin{aligned}
(0,1) \text{ lies on graph of } f(x) = \frac{1}{3}\ln|3x+5| + C \quad &\iff \quad f(0) = 1 \\
&\iff \quad \frac{1}{3}\ln 5 + C = 1 \\
&\iff \quad C = 1 - \frac{\ln 5}{3} \\
&\iff \quad C = \frac{3 - \ln 5}{3}
\end{aligned}$$

Note how this was written down using a *complete mathematical sentence*. The desired function is therefore:

$$\begin{aligned}
f(x) &= \frac{1}{3}\ln|3x+5| + \frac{3 - \ln 5}{3} \\
&= \frac{\ln|3x+5| + 3 - \ln 5}{3}
\end{aligned}$$

EXERCISE 6

♣ 1. Use the Chain Rule to prove that:

$$\frac{d}{dx}\ln|f(x)| = \frac{1}{f(x)}\cdot f'(x)$$

♣ 2. Verify that the function

$$f(x) = \frac{\ln|3x+5| + 3 - \ln 5}{3}$$

has a graph that passes through the point $(0,1)$, and has derivative $f'(x) = \frac{1}{3x+5}$.

EXAMPLE
antidifferentiate

Problem: Antidifferentiate $\dfrac{\ln x}{x}$.

Solution:

$$\begin{aligned}\int \frac{\ln x}{x}\,dx &= \int u\,du \\ &= \frac{u^2}{2} + C \\ &= \frac{1}{2}(\ln x)^2 + C\end{aligned}$$

$u = \ln x$
$du = \frac{1}{x}\,dx$

Check: $\dfrac{d}{dx}(\frac{1}{2}(\ln x)^2) = \frac{1}{2}\cdot 2(\ln x)(\frac{1}{x}) = \dfrac{\ln x}{x}$

EXAMPLE
using a letter different than 'u' for the substitution variable

Problem: Evaluate $\int (2-u)^4\,du$.

Solution: Just use a letter different than 'u' for the substitution variable! Here, the letter 'w' is used.

$$\begin{aligned}\int (2-u)^4\,du &= -\int (2-u)^4(-du) \\ &= -\int w^4\,dw \\ &= -\frac{w^5}{5} + C \\ &= -\frac{1}{5}(2-u)^5 + C\end{aligned}$$

$w = 2 - u$
$dw = -du$

QUICK QUIZ
sample questions

1. What is the idea behind the substitution technique for integration?
2. Solve $\int \frac{1}{2x-1}\,dx$ two ways; without using substitution, and using substitution. Do your answers agree?
3. Where is linearity of the integral used in the substitution technique?
4. Solve: $\int e^{3x}\,dx$
5. Is $\int (3x+\pi)^5\,dx = \frac{(3x+\pi)^6}{18} + C$? Justify your answer.

KEYWORDS
for this section

Transforming a difficult problem into an easier one, the substitution technique for integration, choosing a 'u that works', multiplying by 1 in an appropriate form.

END-OF-SECTION EXERCISES

♣ Evaluate the following indefinite integrals. Be sure to write complete mathematical sentences. Check your answers by differentiating.

1. $\displaystyle\int (2x-1)^{17}\,dx$

2. $\displaystyle\int 5t\sqrt{t^2+3}\,dt$

3. $\displaystyle\int \frac{3\ln 4x}{x}\,dx$

4. $\displaystyle\int (4e^{2t}+e^{1+t})\,dt$

5. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

6. $\displaystyle\int \frac{-1}{2u+5}\,du$

7. $\displaystyle\int \frac{4t+2}{\sqrt{(t^2+t+1)^3}}\,dt$

8. $\displaystyle\int (e^x+1)^5\cdot 3e^x\,dx$

9. Find a function f whose graph passes through the point $(0,4)$, and that has derivative $f'(x)=e^x(e^x+1)^3$.

10. A particle traveling along a line has velocity function given by:

$$v(t)=(t-2)^3$$

It is known that at $t=1$, the particle is at position $\frac{1}{2}$. Find the distance function for this particle.

11. A student passed in the following solution to an integration problem:

$$\begin{aligned}
\int (x^2+1)^5\,dx &= \int \frac{2x}{2x}(x^2+1)^5\,dx \\
&= \frac{1}{2x}\int (x^2+1)^5(2x\,dx) \\
&= \frac{1}{2x}\int u^5\,du \\
&= \frac{1}{2x}\frac{u^6}{6}+C \\
&= \frac{1}{2x}\frac{(x^2+1)^6}{6}+C \\
&= \frac{(x^2+1)^6}{12x}+C
\end{aligned}$$

♣ a) Do you believe that this is a correct solution? If not, where has the student made a mistake?

♣ b) Check the student's solution by finding $\frac{d}{dx}\frac{(x^2+1)^6}{12x}$. (Use the quotient rule.) Is the student's solution correct?