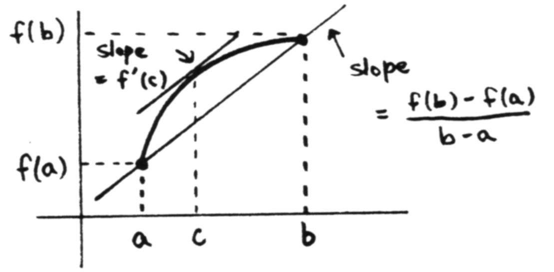
4.9 The Mean Value Theorem

Introduction

The Mean Value Theorem is often referred to as *the Fundamental Theorem of Differential Calculus*. Its importance cannot be overemphasized! Several applications are given in the next section.

The Mean Value Theorem

Suppose that f is differentiable on an open interval (a, b), and continuous on the closed interval $[a, b]$. Then there is at least one number c in (a, b) for which:



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

motivation for the name

The word '*mean*' often has the same mathematical meaning as the word '*average*', and such is the case here. It has been seen that the quotient

$$\frac{f(b) - f(a)}{b - a}$$

represents the *average (mean) rate of change of the function f on* $[a, b]$. Recall that this quotient gives the slope of the line through the points $(a, f(a))$ and $(b, f(b))$. The Mean Value Theorem states that there is at least one number $c \in (a, b)$ where the *instantaneous* rate of change $f'(c)$ is the same as the *average rate of change over the entire interval.*

EXERCISE 1

Consider the function $f\colon [a, b] \to \mathbb{R}$, $f(x) = x^2$. For each interval $[a, b]$ listed below, do the following:

- Sketch the graph of f on $[a, b]$.
- Find $c \in (a, b)$ for which $f'(c) = \frac{f(b)-f(a)}{b-a}$.
- On your graph, show both the tangent line at $(c, f(c))$ and the line through the endpoints of the interval.

♣ 1. $[a, b] = [1, 2]$

♣ 2. $[a, b] = [-1, 1]$

♣ 3. $[a, b] = [-1, 2]$

discussion of the hypotheses to the Mean Value Theorem

The phrase '*f is differentiable on the open interval* (a, b)' means that f is differentiable at x, for every $x \in (a, b)$.

Recall that if f is *differentiable* on (a, b), then it must also be *continuous* on (a, b). (Differentiability is '*stronger*' than continuity!) Thus, by requiring that f be differentiable on (a, b), one is also assured that f is continuous on (a, b). The additional requirement that f be *continuous on the closed interval* $[a, b]$ only adds the assurance that f 'behaves properly' at the endpoints. The following examples illustrate why this requirement is necessary.

EXAMPLE

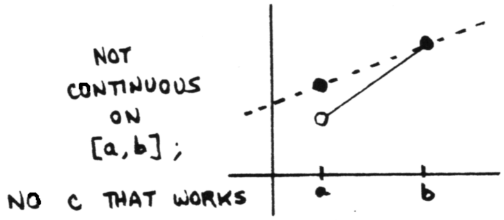
the conclusion of the MVT fails; there is no 'c that works'

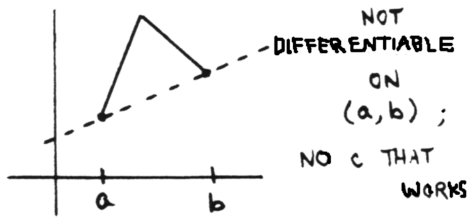
If the hypotheses of the Mean Value Theorem are not met, then its conclusion is not guaranteed.

In the first example below, f is differentiable on (a, b), but not continuous on $[a, b]$.

In the second example, f is not differentible on (a, b).

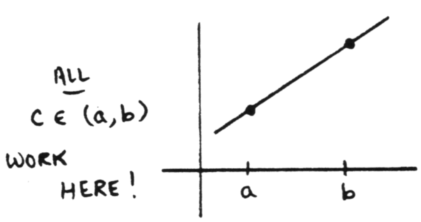
In both cases, there is no '*c that works*'. That is, there is NO POINT $(c, f(c))$ for $c \in (a, b)$ where the tangent line has the same slope as the line through the endpoints of the interval.

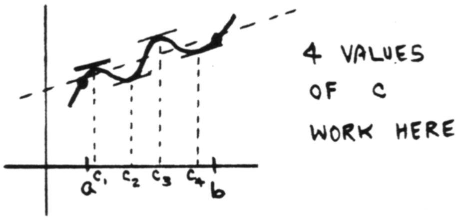




there's not necessarily a unique c that works

The Mean Value Theorem is an *existence* theorem, NOT a *uniqueness* theorem. Thus, it does *not* guarantee a *unique* value of c that works, as the sketches below illustrate.





EXERCISE 2

Sketch the graph of a function f that meets each of the following requirements:

♣ 1. f is differentiable on $(1, 3)$, continuous on $[1, 3]$, $f(3) = 10$, $f(1) = 0$, $f(2) \neq 5$, $f'(2) = 5$, and f is not linear on $[1, 3]$

♣ 2. f is differentiable on $(1, 3)$, continuous on $[1, 3]$, $f(3) = 10$, $f(1) = 0$, $f(2) = 5$, $f'(2) = 5$, and f is not linear on $[1, 3]$

♣ 3. f is differentiable on $(2, 5)$, $f(2) = 1$, $f(5) = 3$, and there is NO $c \in (2, 5)$ for which $f'(c) = \frac{2}{3}$

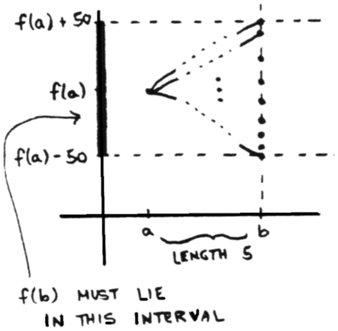
♣ 4. The average rate of change of f on $[0, 2]$ is 4, and yet NOWHERE on $(0, 2)$ does f have an instantaneous rate of change of 4 .

uses of the Mean Value Theorem

The Mean Value Theorem is the tool pulled out in most every situation where *derivative* information is to be used to gain information about the function itself. It is used extensively in analysis for approximate calculations and to obtain error estimates. Some typical uses are presented below. Also, the Mean Value Theorem is used in the next section to obtain some important results.

EXAMPLE

getting bounds on function values



Suppose that f is differentiable on $\mathbb{R}$. Also, suppose it is known that $|f'(x)| \le 10$ for all $x \in \mathbb{R}$. This means that, at any instant, the function values $f(x)$ *never* change at a rate of magnitude greater than 10 units per unit change in x. So if x changes by 1, what is the most that $f(x)$ could change by? Well, it could increase by $10 \cdot 1$. Or, it could decrease by $10 \cdot 1$.

If x changes by 2, what is the most that $f(x)$ could change by? It could increase by $10 \cdot 2$, or decrease by $10 \cdot 2$.

These ideas are made precise by using the Mean Value Theorem. That is, derivative information is used to get a bound on how much the function f can possibly change over any interval $[a, b]$, as follows.

Let $[a, b]$ be any interval. By the Mean Value Theorem, there exists $c \in (a, b)$ for which:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

But, by hypothesis, $|f'(c)| \le 10$. Thus:

$$\left| \frac{f(b) - f(a)}{b - a} \right| \le 10$$

That is:

$$|f(b) - f(a)| \le 10|b - a|$$

For example, suppose that $[a, b]$ is an interval of length 5, so that $|b - a| = 5$. Then, $|f(b) - f(a)| \le 10 \cdot 5$. That is, the distance from $f(b)$ to $f(a)$ must be less than or equal to 50. So, $f(b)$ must lie in the interval $\big(f(a) - 50\,,\ f(a) + 50\big)$.

Bounding techniques such as this are extremely important in analysis.

EXERCISE 3

Suppose f is differentiable on $\mathbb{R}$, and $|f'(x)| \le 2$ for all $x \in \mathbb{R}$. Answer the following questions.

♣ 1. Suppose $f(1) = 5$. In what interval must $f(2)$ lie?

♣ 2. Suppose $f(1) = 5$. In what interval must $f(3)$ lie?

♣ 3. How much can $f(x)$ change by, whenever x changes by an amount Δx?

consequences of the Mean Value Theorem

All the results listed below are consequences of the Mean Value Theorem. Some of these will be studied more thoroughly in future sections. Some use words that have not yet been defined. These results are listed simply to give you an appreciation for the kind of information that can be gleaned from the Mean Value Theorem (MVT)!

For all these results, assume that f is continuous on $[a, b]$ and differentiable on (a, b).

- If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.
- If $f'(x) = g'(x)$ for all $x \in (a, b)$, then f and g differ by at most a constant on $[a, b]$.
- If $f'(x) \ge 0$ for all $x \in (a, b)$ and if $x_1 < x_2$ are in $[a, b]$, then $f(x_1) \le f(x_2)$.
- If $f'(x) > 0$ for all $x \in (a, b)$ and if $x_1 < x_2$ are in $[a, b]$, then $f(x_1) < f(x_2)$.
- If $f'(x) \ge 0$ for all $x \in (a, a + \delta)$, then $(a, f(a))$ is a relative minimum point of f.
- If $f'(x) \ge 0$ for all $x \in (b - \delta, b)$, then $(b, f(b))$ is a relative maximum point of f.

sample proof

Here, the Mean Value Theorem is used to prove the first result in the previous list:

Suppose f is continuous on $[a,b]$ and differentiable on (a,b). If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.

Proof. Let $K = f(a)$. It will be shown that $f(x) = K$ for all $x \in [a,b]$, so that f is constant on $[a,b]$.

Choose any point $x \in (a,b]$. Since f is differentiable on the subinterval (a,x) (why?) and continuous on $[a,x]$ (why?), there exists $c \in (a,x)$ for which:

$$f'(c) = \frac{f(x) - f(a)}{x - a} = \frac{f(x) - K}{x - a}$$

Since $f'(c) = 0$, it must be that

$$\frac{f(x) - K}{x - a} = 0 \ ,$$

so that $f(x) - K = 0$, and thus $f(x) = K$. Thus, f is constant on $[a,b]$. ∎

★★

the proof of the MVT

The proof of the Mean Value Theorem is nontrivial. It often goes like this:

- First, prove that if f is differentiable at x and $f'(x) > 0$, then

  $$f(x - h) < f(x) < f(x + h)$$

  for all positive h sufficiently small. Prove the similar result for $f'(x) < 0$.
- Prove *Rolle's Theorem*: Let f be differentiable on (a,b) and continuous on $[a,b]$. If $f(a) = f(b) = 0$, then there exists at least one number $c \in (a,b)$ for which $f'(c) = 0$.

  Rolle's Theorem is a special case of the Mean Value Theorem.
- To prove the Mean Value Theorem, apply Rolle's Theorem to the function:

  $$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right]$$

the mathematical phrase, 'for all'

The remainder of this section deals with the mathematical phrase, 'for all'. Fortunately, the conventional English usage of this phrase agrees very nicely with its mathematical meaning; for this reason, the author has been able to avoid a careful discussion up to this point. It is time, however, to make things precise.

$S(x)$ denotes a sentence involving the variable x

Let $S(x)$ denote a sentence involving the variable x. For example, $S(x)$ might represent the sentence '$x = 3$', or it might represent the sentence '$3x - 2 > 0$'. Let $\mathcal{U}$ denote the universal set for the variable x.

In order for the sentence

$$\text{' For all } x \in \mathcal{U}, \ \ S(x) \text{ '} \qquad (*)$$

to be true, $S(x)$ must be true, no matter what choice of x is made from the universal set. If there is at least one value of $x \in \mathcal{U}$ for which $S(x)$ is false, then sentence (*) is false.

EXAMPLES

The sentence

$$\text{'For all } x \in \mathbb{R},\ \ x^2 \ge 0\text{ '}$$

is true. No matter what real number x is chosen, the sentence '$x^2 \ge 0$' is true.

The sentence

$$\text{' For all } x \in \mathbb{R},\ \ x^2 > 0\text{ '}$$

is false. Choosing $x = 0$, the sentence '$0^2 > 0$' is false.

The sentence

$$\text{' For all } x \in \mathbb{R},\ \ x^2 < 0\text{ '}$$

is false. Here, no matter *what* value of x is chosen, the sentence '$x^2 < 0$' is false.

The sentence

$$\text{' For all } x > 0,\ \ |x| = x\text{ '}$$

is true. Whenever x is a positive number, the sentence '$|x| = x$' is true.

The sentence

$$\text{' For all sets } A \text{ and } B,\ \ A \subset A \cup B\text{ '}$$

is true. No matter what sets are chosen for A and B, A is always a subset of $A \cup B$.

What does it mean for (*) to be false?

If a sentence of the form

$$\text{' For all } x \in \mathcal{U},\ \ S(x)\text{ '}$$

is false, then all that can be said (without additional information) is that *there is at least one* $x \in \mathcal{U}$ *for which* $S(x)$ *is false.*

EXERCISE 4

TRUE or FALSE:

♣ 1. For all $x \in \mathbb{R}$, $|x| \ge 0$

♣ 2. For all $x \in \mathbb{R}$, $|x| > 0$

♣ 3. For all $t < 0$, $|t| = -t$

♣ 4. For all $x \in (2, 3)$, $x \ge 0$

♣ 5. For all sets A and B, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

♣ 6. For all functions f and g that are differentiable at x, $(f + g)'(x) = f'(x) + g'(x)$

the universal set is sometimes omitted

Sometimes, mathematicians get a bit casual with their use of 'for all'. For example, the universal set is frequently omitted, if it is understood from context.

For example, in a course such as this, where the universal set is understood to be $\mathbb{R}$ (unless otherwise stated), the (true) sentence

$$\text{' For all } x,\ \ |x| \ge 0\text{ '}$$

is understood to be an abbreviation for the more correct sentence:

$$\text{' For all } x \in \mathbb{R},\ \ |x| \ge 0\text{ '}$$

the words ‘for all’ are sometimes omitted

Even more annoying—the words ‘for all’ are often omitted, in certain types of situations, if they are understood from context! For example, the (true) sentence

$$‘\ 2x+1=0 \quad \iff \quad x=-\frac{1}{2}\ ’$$

is really an abbreviation for the more correct (true) sentence:

$$‘\ \text{For all } x\in\mathbb{R},\ \ (2x+1=0 \iff x=-\frac{1}{2})\ ’$$

The connective ‘ $\iff$ ’ is defined via the truth table given below:

A	B	A ⇔ B
T	T	T
T	F	F
F	T	F
F	F	T

Note that the sentence ‘$A \iff B$’ is true precisely when A and B have the same truth values (either they are both true, or both false). Thus, the sentence

$$‘\ \text{For all } x\in\mathbb{R},\ \ (2x+1=0 \quad \iff \quad x=-\frac{1}{2})\ ’$$

is true because, no matter what real number is chosen, the sentences ‘$2x+1=0$’ and ‘$x=-\frac{1}{2}$’ always have the same truth values. Observe that this is in perfect agreement with earlier discussions of mathematical equivalence.

EXERCISE 5

How might a mathematician abbreviate the following (true) sentences, if appropriate information is understood from context?

♣ 1. For all $x\in\mathbb{R}$ and $y\in\mathbb{R}$, $x+y=y+x$

♣ 2. For all $x\in\mathbb{R}$, $x=2 \iff 3x=6$

♣ 3. For all sets A and B, $A\subset A\cup B$

♣ 4. For all statements P and Q, $\big((P\Rightarrow Q) \iff (\text{not } Q \Rightarrow \text{not } P)\big)$

(A *statement* is merely a sentence that is either true, or false, but not both. For example, ‘$1=2$’ is a false statement; ‘$2=1+1$’ is a true statement; and ‘$x=1$’ is not a statement until a particular value of x is substituted into the equation.)

QUICK QUIZ

sample questions

1. Give a precise statement of the Mean Value Theorem.
2. What does the word ‘mean’ in the Mean Value Theorem refer to?
3. Let $f(x)=x^3$ and $[a,b]=[1,3]$. Find the number c that is guaranteed by the Mean Value Theorem. Make a sketch that illustrates your work.
4. Suppose that f is differentiable on (a,b), but there is no $c\in(a,b)$ with $f'(c)=\frac{f(b)-f(a)}{b-a}$. What can be said about the function f?
5. Suppose that f is continuous on $[a,b]$, but there is no $c\in(a,b)$ with $f'(c)$ equal to the average rate of change of f over $[a,b]$. What (if anything) can be said about the function f?

KEYWORDS

for this section

The Mean Value Theorem (MVT), motivation for the name, using the MVT to bound function values, some consequences of the MVT, the mathematical phrase ‘for all’, truth table for $A \iff B$.

END-OF-SECTION EXERCISES

These exercises review many of the ideas in Chapter 4.

1. What information does the limit $\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ give (when it exists)? Answer in English.

2. What information does the limit $\displaystyle\lim_{y\to x}\frac{f(y)-f(x)}{y-x}$ give (when it exists)? Answer in English.

3. Suppose that, for a given function f, it is known that $f'(2) = 4$. What does this tell us about the function f? Sketch the graphs of two different functions satisfying this requirement.

4. Suppose that, for a given function f, it is known that $f(2) = 1$ and $f'(2) = -1$. Sketch the graphs of two different functions satisfying this requirement.

5. Use the *definition of derivative* to find $f'(x)$ if $f(x) = -x^2$.

6. Use the *definition of derivative* to find $f''(x)$ if $f'(x) = 3x$.

7. Sketch the graph of a function f that is continuous at 3, but not differentiable at 3.

8. Is it possible to sketch the graph of a function that is differentiable at 3, but not continuous at 3? Why or why not?

9. Differentiate $f(x) = xe^{2x}\ln(2-x)$. Use any appropriate tools. Then, find: $\mathcal{D}(f)$, $\mathcal{D}(f')$, the equation of the tangent line when $x = 0$.

10. Let $f(x) = x^2$ and $g(x) = \frac{1}{x}$. Find $\frac{d}{dx}f(g(x))$ in two different ways (using the Chain Rule, and NOT using the Chain Rule).