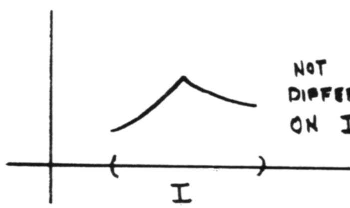
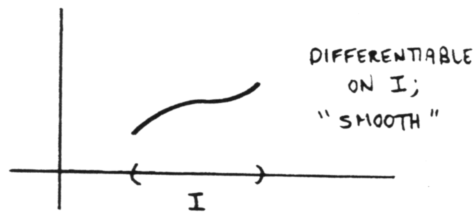
4.7 Higher Order Derivatives

Introduction; smooth functions

When a function f is differentiated, another function, f', is obtained. This *new* function f' may itself be differentiable. Thus, in many cases, one may continually repeat the differentiation process, obtaining the so-called *higher-order derivatives*. This section presents the notation for higher-order derivatives.

If the graph of a function f has a *kink* at x, then f is not differentiable at x. Thus, if f is differentiable at every point in some interval, it must not have any *kinks* in this interval. In this sense, a differentiable function is *smooth*. Mathematicians use the word '*smooth*' to describe the differentiability of a function, but the usage is not entirely consistent: to some, '*smooth*' means once-differentiable; to others, '*smooth*' means infinitely differentiable. In general, the more times a function is differentiable, the 'smoother' it is.





higher-order derivatives; notation *f', f'', f''', $f^{(4)}, \ldots, f^{(n)}$*

The following *prime notation* is used for the higher-order derivatives:

differentiate f to get f';	f' is the (first) derivative of f
differentiate f' to get f'';	f'' is the second derivative of f
differentiate f'' to get f''';	f''' is the third derivative of f
differentiate f''' to get $f^{(4)}$;	$f^{(4)}$ is the fourth derivative of f
differentiate $f^{(4)}$ to get $f^{(5)}$;	$f^{(5)}$ is the fifth derivative of f
	$\vdots$
differentiate $f^{(n-1)}$ to get $f^{(n)}$;	$f^{(n)}$ is the n^{th} derivative of f

The notation f'' can be read either as 'f double prime', or as 'the second derivative of f'.

It gets unwieldy to count the number of prime marks, so it is conventional to change to a *numerical* superscript, in parentheses, from about the fourth derivative on. The notation $f^{(4)}$ is usually read as 'the fourth derivative of f'. Observe that the *name* of the n^{th} derivative is $f^{(n)}$; this function, evaluated at x, is denoted by $f^{(n)}(x)$.

The functions f'', f''', $f^{(4)}, \ldots$ are called the *higher-order derivatives of* f.

infinitely differentiable

If a function f has the property that $f^{(n)}$ exists (and has the same domain as f) for *all* positive integers n, then we say that f is *infinitely differentible.*

EXERCISE 1

What is the prime notation for each of the following?

♣ 1. the second derivative of g

♣ 2. the second derivative of g, evaluated at x

♣ 3. the derivative of f'''

♣ 4. the second derivative of $f^{(6)}$, evaluated at 3

EXAMPLE

Let $P(x) = 2x^5 - x^4 + 2x - 1$. Then:

$$\begin{aligned}
P'(x) &= 10x^4 - 4x^3 + 2 \\
P''(x) &= 40x^3 - 12x^2 \\
P'''(x) &= 120x^2 - 24x \\
P^{(4)}(x) &= 240x - 24 \\
P^{(5)}(x) &= 240 \\
P^{(n)}(x) &= 0 \ , \quad \text{for } n \ge 6
\end{aligned}$$

EXERCISE 2

♣ Find *all* derivatives of:

$$P(x) = 2x^7 - x^3 + 4$$

Be sure to write complete mathematical sentences.

It's a good exercise to differentiate an *arbitrary* polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \ ,$$

since this exercise offers an opportunity to introduce some important *summation* and *factorial* notation. So this is our next project. First, summation notation is introduced.

summation notation;
$\sum_{j=s}^{e} a_j$
the index of the sum is a dummy variable

Summation notation gives a convenient way to display a sum, when the terms share some common property.

For nonnegative integers s ('start') and e ('end') with $s < e$, one defines:

$$\sum_{j=s}^{e} a_j := a_s + a_{(s+1)} + \cdots + a_{(e-1)} + a_e$$

The symbol $\sum_{j=s}^{e} a_j$ is read as: *the sum, as j goes from s to e, of a_j*.

In particular, if $s = 1$ and $e = n$ one gets:

$$\sum_{j=1}^{n} a_j = a_1 + a_2 + \cdots + a_{n-1} + a_n$$

The variable j in the above notation is called the *index of the sum*; observe that *once the sum is expanded, this index j no longer appears*. In this sense, it is a *dummy variable*, and we need not be restricted to use of the letter j for this role. Traditionally, the letters i, j, k, m and n are used as indices for summation, precisely because of the strong convention dictating that these letters denote *integer* variables.

When summation notation appears in text (as opposed to in a display), it usually looks like this: $\sum_{j=1}^{n} a_j$. This way, it is not necessary to put extra space between the lines to make room for the '$j = 1$' and 'n'.

EXAMPLE
using summation notation

For example,

$$\sum_{i=3}^{7} a_i = a_3 + a_4 + a_5 + a_6 + a_7$$

and:

$$\sum_{k=2}^{5} (k-3)^k = (2-3)^2 + (3-3)^3 + (4-3)^4 + (5-3)^5$$

Also:

$$\sum_{j=1}^{4} 5 = \overbrace{5}^{j=1} + \overbrace{5}^{j=2} + \overbrace{5}^{j=3} + \overbrace{5}^{j=4} = 4 \cdot 5 = 20$$

The sum

$$1 + 2 + \ldots + 207$$

could be written as:

$$\sum_{k=1}^{207} k \quad \text{or} \quad \sum_{n=1}^{207} n \quad \text{or} \quad \sum_{m=1}^{207} m$$

However, don't write something like $\sum_{i=1}^{207} k$, unless you *really want* the expression below!

$$\sum_{i=1}^{207} k = \overbrace{k + k + \cdots + k}^{\text{207 times!}} = 207k$$

EXERCISE 3

practice with summation notation

♣ 1. Expand the following sums. (You need not simplify the resulting sums.)

$$\sum_{j=1}^{6} b_j \ , \quad \sum_{k=1}^{5} (k+1)^k \ , \quad \sum_{m=0}^{4} (m+1) \ , \quad \sum_{i=1}^{n} 2i$$

♣ 2. Write the sum $\sum_{i=1}^{n} 2i$ using a different index.

♣ 3. Let k be a constant. Prove that:

$$\sum_{j=1}^{n} k a_j = k \sum_{j=1}^{n} a_j$$

(Thus, you can 'slide' constants out of a sum.) Be sure to write complete mathematical sentences.

♣ 4. Write the following sums using summation notation:

$$1 + 2 + 3 + \cdots + 100$$
$$34 + 35 + 36 + \cdots + 79$$
$$2 + 4 + 6 + \cdots + 78$$
$$5^2 + 6^3 + 7^4 + 8^5 + \cdots + 20^{17}$$

♣ 5. Prove the following statement:

$$\frac{d}{dx} \sum_{i=1}^{n} f_i(x) = \sum_{i=1}^{n} f_i'(x)$$

You may assume that the functions f_i are all differentiable at x. Be sure to write complete mathematical sentences, and justify each step of your proof.

polynomials are infinitely differentiable

Now, let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be an arbitrary n^{th} order polynomial (so, $a_n \neq 0$). Using summation notation, one can write:

$$P(x) = \sum_{i=0}^{n} a_i x^i$$

(Recall that $x^0 = 1$.) Differentiating once (and using the fact that the derivative of a sum is the sum of the derivatives) yields:

$$\begin{aligned} P'(x) &= \sum_{i=0}^{n} i \cdot a_i x^{i-1} \\ &= \sum_{i=1}^{n} i \cdot a_i x^{i-1} \end{aligned}$$

The index changed from a starting value of 0 to a starting value of 1 since when $i = 0$ the term $i \cdot a_i x^{i-1}$ vanishes, and hence contributes nothing to the sum. Continuing:

$$\begin{aligned} P''(x) &= \sum_{i=2}^{n} i(i-1)a_i x^{i-2} \\ P'''(x) &= \sum_{i=3}^{n} i(i-1)(i-2)a_i x^{i-3} \\ &\vdots \\ P^{(j)}(x) &= \sum_{i=j}^{n} i(i-1)(i-2)\cdots(i-(j-1))a_i x^{i-j} \quad \text{for } 1 \le j \le n \end{aligned}$$

factorial notation, $k!$

The previous formula for $P^{(j)}$ can be cleaned up a bit by using *factorial notation*, discussed next.

For a positive integer k, one defines:

$$k! := k(k-1)(k-2)\cdots(1)$$

The expression ‘ $k!$ ’ is read as ‘k *factorial*’. By definition, $0! = 1$.

For example: $3! = 3 \cdot 2 \cdot 1 = 6$ and $200! = 200 \cdot 199 \cdot 198 \cdot \ \ldots \ \cdot 2 \cdot 1$

The product $20 \cdot 19 \cdot 18 \cdot \ldots \cdot 5$ can be written in factorial notation, if one first multiplies by 1 in an appropriate form:

$$\begin{aligned} 20 \cdot 19 \cdot 18 \cdot \ \ldots \ \cdot 5 &= 20 \cdot 19 \cdot 18 \cdot \ \ldots \ \cdot 5 \cdot \frac{4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} \\ &= \frac{20 \cdot 19 \cdot 18 \cdot \ \ldots \ \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} \\ &= \frac{20!}{4!} \end{aligned}$$

This technique is used below, in order to ‘clean up’ the expression for $P^{(j)}$.

'cleaning up' the expression for $P^{(j)}$

Using the same 'multiply by 1 in an appropriate form' technique illustrated above, one gets:

$$\begin{aligned} &i(i-1)(i-2)\cdots(i-(j-1)) \\ &\qquad = i(i-1)(i-2)\cdots(i-(j-1))\cdot\frac{(i-j)(i-(j+1))\cdots(1)}{(i-j)(i-(j+1))\cdots(1)} \\ &\qquad = \frac{i!}{(i-j)!} \quad \text{for } i \ge j \end{aligned}$$

Thus, all the derivatives of an arbitrary n^{th} order polynomial P can be expressed as:

$$P^{(j)}(x) = \begin{cases} \displaystyle\sum_{i=j}^{n} \frac{i!}{(i-j)!} a_i x^{i-j} & \text{for } 1 \le j \le n \\ 0 & \text{for } j > n \end{cases}$$

Observe that although this notation is extremely compact, it can (especially for a beginner) make an easy idea seem difficult. For experts, however, the compactness of this notation can be extremely beneficial.

EXERCISE 4

Let $P(x) = \sum_{i=0}^{3} a_i x^i$.

♣ 1. Expand this sum. How many terms does P have?

♣ 2. Show that

$$P'(x) = \sum_{i=1}^{3} i \cdot a_i x^{i-1} ,$$

by expanding the sum, and verifying that it does indeed give a correct formula for P'.

♣ 3. Find formulas for P'' and P''', in summation notation.

♣ 4. What is $P^{(n)}$, for $n \ge 4$?

EXERCISE 5

practice with factorial notation

♣ 1. Express the following numbers as products. It is not necessary to multiply out these products.

$$5!\ , \quad 0!\ , \quad 100!$$

♣ 2. Write the following products using factorial notation:

$$10 \cdot 9 \cdot 8 \cdot \ \ldots\ \cdot 2 \cdot 1$$
$$207 \cdot 206 \cdot 205 \cdot \ \ldots\ \cdot 1$$

♣ 3. Write the following product using factorial notation:

$$105 \cdot 104 \cdot 103 \cdot \ \ldots\ \cdot 50$$

Leibniz notation for higher-order derivatives

Here is the Leibniz notation for higher-order derivatives. Let y be a function of x. Then:

$$\begin{aligned} \frac{d}{dx}(y) &= \frac{dy}{dx} && \text{is the first derivative} \\ \frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d^2y}{dx^2} && \text{is the second derivative} \\ \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) &= \frac{d^3y}{dx^3} && \text{is the third derivative} \\ &\vdots \\ \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) &= \frac{d^ny}{dx^n} && \text{is the } n^{th} \text{ derivative} \end{aligned}$$

If one wishes to emphasize that the derivative $\frac{d^ny}{dx^n}$ is being evaluated at a specific value of x, say $x = c$, then one can write either:

$$\frac{d^ny}{dx^n}(c) \quad \text{or} \quad \frac{d^ny}{dx^n}\Big|_{x=c}$$

At first glance, the lack of symmetry in this notation is disturbing: for example, why should we write $\frac{d^2y}{dx^2}$, and not the more symmetric $\frac{d^2y}{d^2x}$?

However, it should be clear from the process illustrated above why this 'unsymmetry' arises. At the n^{th} step, one 'sees' n 'factors' of d upstairs, hence d^ny. Also, at the n^{th} step, one 'sees' n 'factors' of dx downstairs, hence $(dx)^n$, shortened to the simpler notation dx^n. (After all, it *is only notation*, so we want it to be as simple as possible, without sacrificing clarity.)

EXERCISE 6

What is the Leibniz notation for each of the following?

♣ 1. the second derivative of y (where y is a function of x)

♣ 2. the second derivative of y (where y is a function of t)

♣ 3. the second derivative of g (where g is a function of x)

♣ 4. the second derivative of g, evaluated at 2

♣ 5. the derivative of $\frac{d^3y}{dx^3}$

♣ 6. the second derivative of $\frac{d^3y}{dx^3}$, evaluated at 3

EXERCISE 7

In problems (1) and (2), find the second derivative of the given function. Use any appropriate notation.

♣ 1. $y = \frac{x}{e^x}$

♣ 2. $f(x) = \frac{1}{x-1} + \frac{1}{x-2}$

♣ 3. Find the equation of the tangent line to the graph of the first derivative of $f(x) = \frac{x}{e^x}$ at $x = 0$.

QUICK QUIZ
sample questions

1. What is meant by the phrase, 'the higher derivatives of a function f'?
2. Write the second derivative of f, evaluated at x, using both prime notation and Leibniz notation.
3. Expand the sum: $\sum_{i=1}^{3} i^{i+1}$
4. Write $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ using factorial notation.
5. State that 'the derivative of a sum is the sum of the derivatives', using summation notation.

KEYWORDS
for this section

Smooth functions, higher-order derivatives, prime notation for higher-order derivatives, infinitely differentiable, summation notation, factorial notation, Leibniz notation for higher-order derivatives.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a SENTENCE (SEN).

♣ For any *sentence*, state whether it is TRUE, FALSE, or CONDITIONAL.

1. If f is differentiable at x, then the number $f'(x)$ gives the slope of the tangent line to the graph of f at the point $(x, f(x))$.
2. If f is differentiable at x, then the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists, and gives the slope of the tangent line to the graph of f at the point $(x, f(x))$.
3. $f'(x)$
4. $f'(3)$
5. $f'(x) = 2x$
6. $y' = 3$
7. If f and g are differentiable at x, then $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$.
8. If f is differentiable at c, then $f'(c) = \frac{df}{dx}(c)$.
9. $\ln ab$
10. For $a > 0$ and $b > 0$, $\ln ab = \ln a + \ln b$.
11. $f'(g(x)) \cdot g'(x)$
12. $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$
13. $10 \cdot 9 \cdot 8 \cdot \ldots \cdot 1$
14. $10! = 10 \cdot 9 \cdot 8 \cdot \ldots \cdot 1$
15. $\sum_{i=0}^{3} i = 6$
16. $\sum_{j=1}^{n} a_j$
17. If f is differentiable at c, then $f'(c) = 2$.
18. f is differentiable at c if and only if f is continuous at c