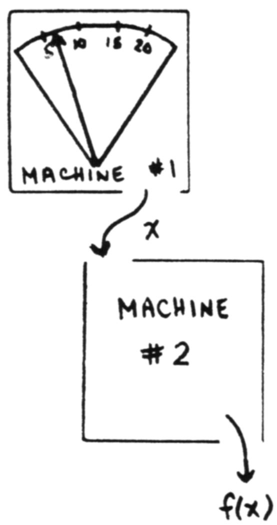
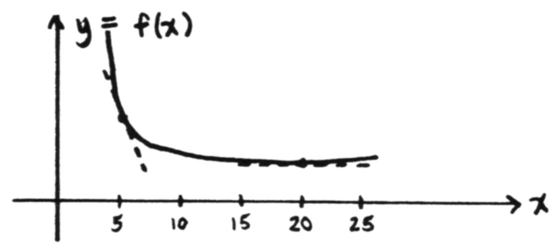
4.4 Instantaneous Rates of Change

Introduction

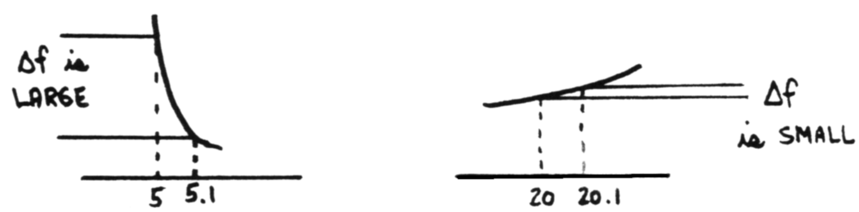
The number $f'(x)$ gives the *slope* of the tangent line to the graph of f at the point $(x, f(x))$ (when the tangent line exists and is not vertical).



Let's think about this information, from a practical viewpoint. Suppose, in a certain laboratory, there are two machines; call them machine 1 and machine 2. Each day, you must take a reading x from machine 1. This reading is then input into machine 2, which produces an output $f(x)$. Suppose that the relationship between the input x and the output $f(x)$ is shown below.



When the input is 20, the slope of the tangent line to the graph of f is of small magnitude. That is, when x changes from 20 by some small amount, the function value will not change very much. So, if you have misread the information from machine 1 slightly, this will not dramatically affect the output from machine 2.

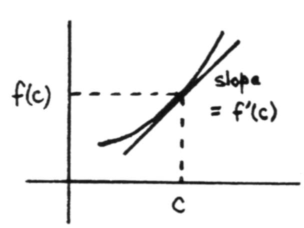


However, when the input is 5, the slope of the tangent line to the graph of f is of large magnitude. Thus, when x changes from 5 by some small amount, the function value will change dramatically. So, if you have misread the information from machine 1 slightly, this *will* dramatically affect the output from machine 2 (a bad situation).

Thus, the information about *how fast the function is changing at a point* can be vitally important.

instantaneous rates of change

There is an important interpretation of the information that $f'(x)$ gives us: $f'(x)$ *tells us how fast the function f is changing at the point* $(x, f(x))$.



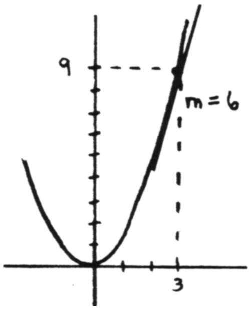
More precisely, for a fixed value of c, the number $f'(c)$ gives the *instantaneous rate of change* of the function values $f(x)$ with respect to x, at the point $(c, f(c))$.

That is, $f(x)$ changes $f'(c)$ times as fast as x at the point $(c, f(c))$.

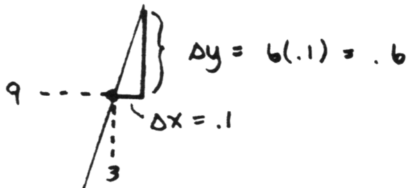
In many situations, we can *use this information to approximate nearby function values,* as illustrated in the next example.

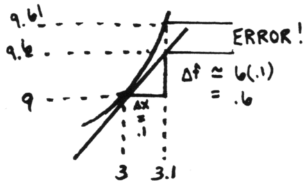
using $f'(x)$ to predict nearby function values

Consider the function $f(x) = x^2$, with derivative $f'(x) = 2x$. The point $(3, 9)$ lies on the graph of f, and the slope of the tangent line at this point is $f'(3) = 2(3) = 6$.



Suppose that knowledge of the function f is lost; *all you now know* is that the point $(3, 9)$ lies on *some graph*, and the slope of the tangent line at this point is 6.





You are asked to *approximate* the function value when $x = 3.1$. This is certainly possible. You know that when $x = 3$, the function values are changing 6 times as fast as the x values. So, if x changes by some small amount, it is reasonable to expect that $f(x)$ will change by approximately 6 times this amount.

The change in x from $x = 3$ to $x = 3.1$ is $\Delta x = 0.1$. So we expect $f(x)$ to change by approximately $6(\Delta x) = 6(0.1) = 0.6$. Thus, it is reasonable to approximate the *new* function value by the *old* function value, plus 0.6. Thus, $f(3.1) \approx 9 + 0.6 = 9.6$.

Now, you find the missing paper and remember that $f(x) = x^2$. Thus, it is now possible to compute the *actual value* of the function when $x = 3.1$: $f(3.1) = (3.1)^2 = 9.61$. How far off were you? You had *estimated* the value at 9.6; the actual value was 9.61. Not bad!

So *we can use the information about the value of the derivative <u>at a single point</u> to approximate values of the function that are nearby!*

the slopes of the tangent lines are changing as we move from point to point

Observe that the approximation we got in the previous example was just that—*an approximation.* That is because our answer was based on the fact that the slope of the tangent line at the point $(3, 9)$ is 6; but *as soon as we move away from that point, this is no longer true.* Indeed, the slopes of the tangent lines *increase* as we travel from $x = 3$ to $x = 3.1$; they increase from 6 to 6.2. So, actually, the rate of change of the function is *faster than* 6 over the interval from $x = 3$ to $x = 3.1$. This is why our approximation of 9.6 was a bit low. The actual function value is 9.61.

EXERCISE 1

Suppose that all you know about a function f is that the point $(3, 7)$ lies on the graph, and the slope of the tangent line at this point is 5.

♣ 1. Approximate, as best you can, $f(3.2)$ and $f(2.9)$.

♣ 2. Sketch two curves that satisfy $f(3) = 7$ and $f'(3) = 5$. On your sketches, show your *approximation* to $f(3.2)$, and the *actual value* $f(3.2)$.

♣ 3. Suppose you now learn that $f(x) = x^2 - x + 1$. Verify that the point $(3, 7)$ lies on the graph of f, and that the slope of the tangent line here is 5.

♣ 4. How far off were your estimates? That is, compare the actual values of $f(3.2)$ and $f(2.9)$ to your estimates from (1).

★★
f' must be continuous

An underlying assumption in this scheme is that f' is continuous in the interval about x under investigation. It is of course possible for a function f to be differentiable at x, and yet have f' NOT be continuous at x. Take, for example:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

This function has as its derivative:

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

So, f is differentiable at 0 and $f'(0) = 0$. However, f' is not continuous at 0.

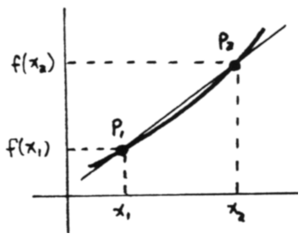
In a motivated class, this importance of the *continuity of f'* could be discussed. Perhaps note that, in analysis, the class of functions that are both *differentiable* on a set S AND have the property that *f' is continuous on S* are given a special name, $C^1(S)$, due to their importance!

DEFINITION
average rate of change

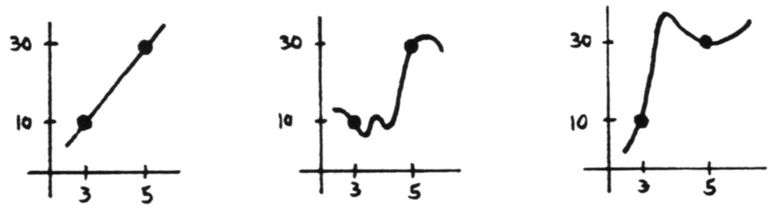
Given a function f and two points $P_1 = (x_1, f(x_1))$, $P_2 = (x_2, f(x_2))$ on the graph of f, we define:

$$\text{the } \textit{average rate of change of } f \textit{ from } x_1 \textit{ to } x_2 := \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Thus, the *average rate of change of f from x_1 to x_2* represents the slope of the secant line through P_1 and P_2.



This seems entirely reasonable: if the points are $(3, 10)$ and $(5, 30)$, then the function has changed by 20 when x has changed by 2, and it seems reasonable to say that, on average, the function has changed by $\frac{20}{2}$ (per a unit change in x). Of course, as illustrated below, the function may behave *entirely differently* between these two points, and yet still exhibit the same average rate of change.



$\Delta f := f(x_2) - f(x_1)$
$\Delta x := x_2 - x_1$
average ROC $= \frac{\Delta f}{\Delta x}$

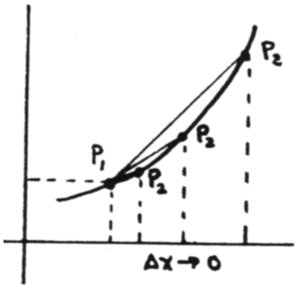
Letting Δf denote the change in function values $f(x_2) - f(x_1)$, and Δx denote the change in x-values $x_2 - x_1$, one can write:

$$\text{average rate of change of } f = \frac{\Delta f}{\Delta x}$$

as $\Delta x \to 0$, the average ROC approaches the instantaneous ROC

Suppose that, for a given function f, there IS a tangent line at the point P_1. If we fix this point P_1, and let the second point P_2 slide closer and closer to P_1 (thus letting $\Delta x \to 0$), then the secant line through P_1 and P_2 approaches the tangent line at P_1. In words, the *average rate of change approaches the instantaneous rate of change, as Δx approaches* 0 .

further appreciation for the Leibniz notation



Whereas the notation Δx is used to denote a *finite* change in x (say from $x = 3$ to $x = 3.1$), it is common in calculus to let (intuitively) dx denote an *infinitesimal* change in x. That is, somehow, dx is meant to represent an *arbitrarily small* change in x.

Similarly, df is used to denote an *arbitrarily small* change in function values.

Armed with this intuition, we can gain a further appreciation for the Leibniz *notation* for the derivative: As Δx approaches 0, $\frac{\Delta f}{\Delta x}$ approaches the slope of the tangent line at x. In general, the closer Δx is to 0, the closer $\frac{\Delta f}{\Delta x}$ will be to the slope of the tangent line at x. The Leibniz notation $\frac{df}{dx}$, therefore, is meant to connote the image of an *infinitesimal change in f* divided by an *infinitesimal change in x*.

More precisely, of course, the notation $\frac{df}{dx}$ should conjure the image of Δx going to 0: it should conjure up the *process* of the second point sliding ever closer to the first. If the notation $\frac{df}{dx}$ succeeds in reminding you of this process each time you see it, then the notation is a good notation.

EXERCISE 2

For the function $f(x) = x^3$, find the average rate of change of f from:

♣ 1. $x = 1$ to $x = 2$

♣ 2. $x = 1$ to $x = 1.5$

♣ 3. $x = 1$ to $x = 1.2$

♣ 4. Find the instantaneous rate of change at $x = 1$. Compare with the average rates of change you just found, and comment.

♣ 5. Why were all of the average rates of change *higher* than the instantaneous rate of change?

EXERCISE 3

For the function $f(x) = -x^2$, find the average rate of change of f from:

♣ 1. $x = -2$ to $x = -1$

♣ 2. $x = -2$ to $x = -1.5$

♣ 3. $x = -2$ to $x = -1.8$

♣ 4. Find the instantaneous rate of change at $x = -2$. Compare with the average rates of change you just found, and comment.

♣ 5. Why were all of the average rates of change *lower* than the instantaneous rate of change?

EXERCISE 4

♣ 1. Sketch the graph of a function f that satisfies the following properties:

- The average rate of change from $x = 0$ to $x = 1$ is 5.
- The instantaneous rate of change at $x = 0$ is -1 and the instantaneous rate of change at $x = 1$ is 2.
- $f(0.5) = 6$

♣ 2. Now, sketch a different curve that satisfies the same properties.

relationship between differentiability and continuity

This section is closed with a very important theorem, stating a relationship between differentiability and continuity.

THEOREM
differentiable at x implies continuous at x

If a function is *differentiable* at x, then it is *continuous* at x.

differentiability is 'stronger' than continuity

One often refers to this fact by saying that *differentiability is a stronger condition than continuity.* That is, requiring a tangent line to exist at a point, forces the function to be continuous at that point.

proving an implication

This theorem is an implication; that is, it is of the form '*If A, then B*'. Remember that a sentence of this form is automatically true whenever A is false; in such cases, it is called *vacuously true.* To verify that the sentence is *always* true, then, we need only verify that whenever A is true, so is B.

direct proof of $A \implies B$

The proof of an implication '*If A, then B*' often takes the following form:

HYPOTHESIS:	Suppose A is true.
BODY OF PROOF:	Use the fact that A is true (and other necessary tools) to show that B is true.
CONCLUSION:	Conclude that B is true.

This form of proof, where we assume that A is true and then show that B must also be true, is called a *direct proof* of $A \implies B$.

In preparation for the proof of the preceding theorem, the next exercise addresses equivalent characterizations of continuity.

EXERCISE 5
equivalent characterizations of continuity at x

Recall that, by definition:

$$f \text{ is continuous at } c \iff \lim_{x\to c} f(x) = f(c)$$

This limit statement makes precise the following intuition: whenever the inputs to f are close to c, the corresponding outputs are close to the number $f(c)$.

♣ 1. What is the *dummy variable* in the limit statement $\lim_{x\to c} f(x) = f(c)$?

♣ 2. Rewrite $\lim_{x\to c} f(x) = f(c)$ with dummy variable y.

♣ 3. Now, using dummy variable y, write the limit statement corresponding to the sentence: *f is continuous at x*.

♣ 4. Convince yourself that the following sentences are all equivalent ways to say that 'f is continuous at x':

$$\begin{aligned} f \text{ is continuous at } x &\iff \lim_{y\to x} f(y) = f(x) \\ &\iff \lim_{h\to 0} f(x+h) = f(x) \\ &\iff \lim_{h\to 0}\big(f(x+h) - f(x)\big) = 0 \end{aligned}$$

For example, if the sentence $\lim_{h\to 0} f(x+h) = f(x)$ is true, then when h is close to 0, $f(x+h)$ must be close to $f(x)$. But when h is close to 0, $x+h$ is close to x. So this says that when the inputs are close to x, the corresponding outputs must be close to $f(x)$, as desired.

One of these equivalent characterizations is used in the next proof.

PROOF
that f differentiable at x implies f continuous at x

Proof. Suppose that f is differentiable at x. That is,

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists, and is given the name $f'(x)$.

BODY OF PROOF

To show that f is *continuous* at x, it is shown equivalently that:

$$\lim_{h\to 0}\big(f(x+h)-f(x)\big) = 0$$

To this end:

$$\begin{aligned}\lim_{h\to 0}\big(f(x+h)-f(x)\big) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\cdot h && (\text{for } h\neq 0,\ \frac{h}{h}=1)\\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\cdot \lim_{h\to 0} h && (\text{property of limits})\\ &= f'(x)\cdot 0\\ &= 0\end{aligned}$$

CONCLUSION

Thus, f is continuous at x. ∎

EXERCISE 6

♣ 1. What is the *hypothesis* of the theorem just proved?

♣ 2. Where was this hypothesis used in the previous proof?

short form of the previous proof

As mathematicians get more and more proficient at writing proofs, typically the proofs become shorter and shorter. The previous result could be proven more briefly as follows:

Proof. Let f be differentiable at x. Then

$$\lim_{h\to 0} f(x+h)-f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\cdot h = f'(x)\cdot 0 = 0. \quad ∎$$

Observe that *all the excess* has been cut out of this proof; only the hypothesis and the 'heart' of the body of the proof remain.

the contrapositive of the previous theorem

The previous result is an implication:

IF f is differentiable at x, THEN f is continuous at x. (1)

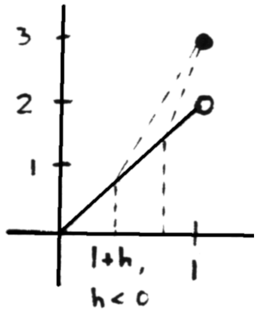
The *contrapositive* of this implication is:

If f is *not* continuous at x, then f is *not* differentiable at x. (2)

Since an implication is equivalent to its contrapositive, and since (1) is true (♣ Why?), sentence (2) is also true. Thus, whenever a function f is NOT continuous at x, we can conclude that f is NOT differentiable at x. This often gives an elegant way to prove that a function is not differentiable at a point, as illustrated next.

EXAMPLE

not continuous $\implies$ not differentiable



Consider the function $f\colon [0,1] \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} 2x & x \in [0,1) \\ 3 & x = 1 \end{cases}$$

Since f is *not* continuous at $x = 1$, it is *not* differentiable at $x = 1$.

The fact that f is not differentiable at $x = 1$ could also be proven directly: the limit

$$\begin{aligned} \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} &= \lim_{h\to 0^-} \frac{2(1+h) - 3}{h} \\ &= \lim_{h\to 0^-} \frac{2h - 1}{h} \\ &= \lim_{h\to 0^-} 2 - \frac{1}{h} \end{aligned}$$

does not exist.

However, citing the previous result is more elegant.

QUICK QUIZ

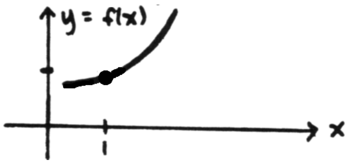
sample questions

1. Let $f(x) = x^3$. Find the average rate of change of f from $x = 1$ to $x = 2$. What is the graphical interpretation of this number?
2. Let $f(x) = x^3$. Find the instantaneous rate of change of f at $x = 1$. What is the graphical interpretation of this number?
3. Consider the function f graphed below. You are not given enough information to find average or instantaneous rates of change. However, you can answer the following question:

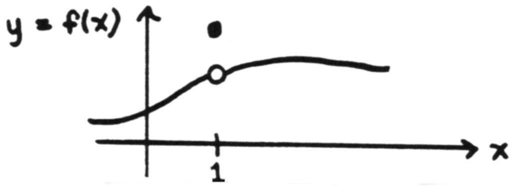
   the instantaneous rate of change of f at $x = 1$ is

   (circle one) (less than greater than equal to)

   the average rate of change of f from $x = 1$ to $x = 2$.

4. Sketch the graph of a function f that satisfies the following properties: $f(x) < 0$ for all $x \in [1,3]$; $f(1) = -5$; the average rate of change of f from $x = 1$ to $x = 3$ is 2; and $f'(2) = -1$.
5. Prove that the function f shown below is not differentiable at $x = 1$.

KEYWORDS

for this section

Instantaneous rate of change, using $f'(x)$ to predict nearby function values, average rates of change, relationship between the instantaneous and average rates of change, What process should the Leibniz notation $\frac{df}{dx}$ conjure up?, relationship between differentiability and continuity, direct proof of $A \implies B$, equivalent characterizations of continuity.

END-OF-SECTION EXERCISES

♣ In each question below, you are given a *point* on the graph of a function f, and the *instantaneous rate of change* of the function at this point.

♣ Use this limited information to predict the value of f at the given nearby point.

♣ Make a sketch that illustrates what you are doing.

1. point: $(1,3)$
   instantaneous ROC at this point: 2
   nearby point: $(2,?)$

2. point: $(2,5)$
   instantaneous ROC at this point: -1
   nearby point: $(3,?)$

3. point: $f(3)=-1$
   instantaneous ROC at this point: $f'(3)=5$
   nearby point: $x=4$

4. point: $f(-3)=2$
   instantaneous ROC at this point: $f'(-3)=1$
   nearby point: $x=-4$.