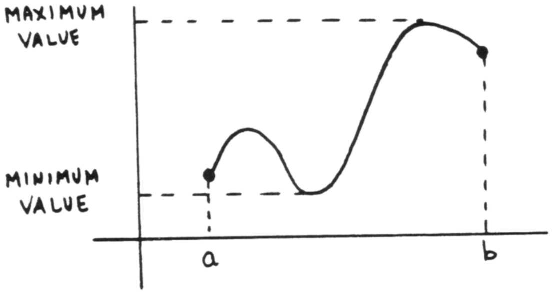
3.7 The Max-Min Theorem

Introduction

This section presents a second fundamental property of functions that are continuous on a closed interval. Roughly, the *Max-Min Theorem* says that a function continuous on $[a, b]$ must attain both a maximum and minimum value on this interval.

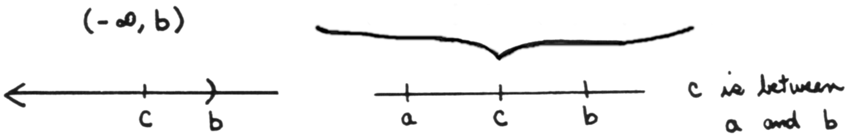
We begin with a discussion of maximum and minimum values on an interval.



interval I

In the next definition, I is an interval of real numbers containing c. For example, I may be of any of these forms:

(a, b) $[a, b]$ $(a, b]$ $[a, b)$ $(a, c]$ $[a, c]$ $[c, b)$ $[c, b]$



DEFINITION

minimum of f on I; maximum of f on I; extreme values of f on I

Let f be defined on an interval I containing c.

The number $f(c)$ is a *minimum (value) of f on I* $\iff$ $f(c) \le f(x) \ \ \forall\, x \in I$

The number $f(c)$ is a *maximum (value) of f on I* $\iff$ $f(c) \ge f(x) \ \ \forall\, x \in I$

When such maximum or minimum values do occur, they are called *extreme values of f on I*. Note that a 'value' is a *number*.

One is usually interested not only in the number $f(c)$ but also the place or places where this number occurs. Such a point $(c, f(c))$ is called an *extreme (maximum or minimum) point* of f on I.

interpreting this definition

This definition assigns meaning to the phrase '$f(c)$ *is a minimum of* f *on* I'. The assigned meaning is this: $f(c) \le f(x) \ \ \forall\, x \in I$. That is, no matter what value of x is chosen from I, it must be that $f(c) \le f(x)$. Thus, $f(c)$ is the least number taken on by f over the interval I.

The definition can also be used 'from right to left'. That is, if it is known that $f(c) \le f(x) \ \ \forall\, x \in I$, then, by this definition, $f(c)$ is a minimum of f on I.

Definitions are *always* statements of *equivalence*. This definition states that the two sentences

$$f(c) \text{ is a minimum of } f \text{ on } I$$

and

$$f(c) \le f(x) \ \ \forall\, x \in I$$

are *equivalent*, and hence can be used interchangeably.

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every definition is (either implicitly or explicitly) a statement of equivalence

Every definition is a statement of equivalence. Since mathematicians know this fact, they often get a bit sloppy about how they state definitions. It is common to see things like this:

DEFINITION. If object x has property P, then x is called a *glob*.

Or,

DEFINITION. The object x is called a *glob* if it has property P.

What the author really means here is:

DEFINITION. x has property $P \iff x$ is a glob

So: if x has property P, then it is a glob. And, if x is a glob, then x has property P. The two sentences are interchangeable.

That is, although definitions are commonly stated as sentences of the form '*If A, then B*', they are ALWAYS really statements of equivalence.

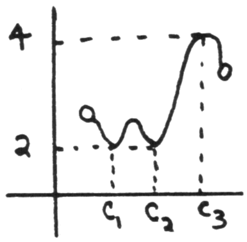
This is NOT true of theorems, however!

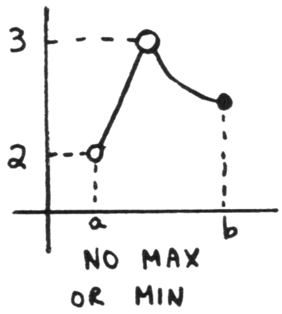
extreme values may or may not occur

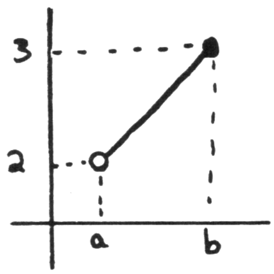
The following examples show that extreme values on an interval I may or may not exist.

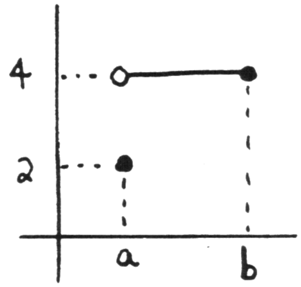
In the first sketch below, the minimum value of f on $I := (a, b)$ is 2, and is attained in two places; $f(c_1) = f(c_2) = 2$. Thus, $(c_1, 2)$ and $(c_2, 2)$ are both minimum points of f on I. Also, the maximum value of f on I is 4; $(c_3, 4)$ is a maximum point of f on I.

In the second sketch, take I to be the interval $(a, b]$. There is no minimum value. The number 2 is 'trying' to be the minimum value, but is never taken on. That is, there is no $c \in I$ with $f(c) = 2$. The only outputs taken on are those in the interval $(2, 3)$: does this set $(2, 3)$ have a least element? No! One can 'reach into' the output pile $(2, 3)$ and choose a number as close to 2 as desired; and then reach in again and choose a number even closer to 2. Since the number 2 is NOT in this pile, there is no least element. There is also no maximum value. ♣ Why?









In the third sketch, take $I := (a, b]$. The maximum value of f on I is 3; the point $(b, 3)$ is a maximum point. There is no minimum value.

In the last sketch, take $I := [a, b]$. The minimum value is 2; the point $(a, 2)$ is the only minimum point. The maximum value is 4, and is attained (taken on) by every $x \in (a, b]$. That is, the points $(x, 4)$ are all maximum points, for every $x \in (a, b]$.

Observe, in all these examples, that whenever a maximum or minimum value FAILS to exist, it is due either to a discontinuity of the function, or a missing endpoint.

EXERCISE 1

practice with extreme values

For each of the following, make a sketch illustrating a function f and an interval I satisfying the stated requirements:

♣ 1. I is an open interval, f is continuous at every point in I, 3 is the minimum value on I, there is no maximum value

♣ 2. I is neither open nor closed, f is not continuous at every point in I, -1 is the minimum value on I, 2 is the maximum value on I

♣ 3. f is defined on $[a, b]$, $\lim_{x \to a^+} f(x) = 2$, the minimum value of f on I is 0, the maximum value of f on I is 2

EXERCISE 2
minimum values versus minimum points

♣ 1. If a function f has a minimum value on I, must this minimum value be unique? That is, can there be two different numbers, both of which are minimum values on I?

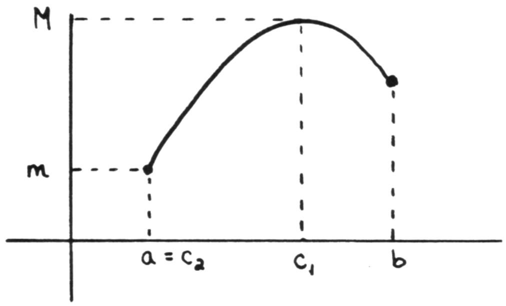
♣ 2. If a function f has a minimum point on I, must this point be unique? Or, can there be more than one point where the minimum value is attained?

conditions under which extreme values will always exist

The next theorem tells us that if a function is *continuous* on a *closed interval*, then it *must* take on both a maximum and minimum value on this interval.

THEOREM
the Max-Min Theorem

If a function f is continuous on a closed interval $[a, b]$, then f must take on both a maximum value M and a minimum value m on $[a, b]$. That is, there must exist $c_1 \in [a, b]$ for which $f(c_1) = M$. Also, there must exist $c_2 \in [a, b]$ for which $f(c_2) = m$.



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idea of proof of the Max-Min Theorem

To prove the Max-Min Theorem, one first shows that every continuous function on a closed interval is bounded on this interval. Let M be the least upper bound of the set $\{f(x) \mid x \in [a, b]\}$, and define:

$$g(x) := \frac{1}{M - f(x)}$$

Argue by contradiction. If f does NOT take on the value M, then g is continuous on $[a, b]$, and hence must be bounded on $[a, b]$. But, g is NOT bounded on $[a, b]$, since in this case $f(x)$ must take on values arbitrarily close to M. This provides the desired contradiction.

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a more general topological result

The Max-Min Theorem is a special case of an extremely important topological theorem: every continuous function on a compact set attains both a maximum and a minimum.

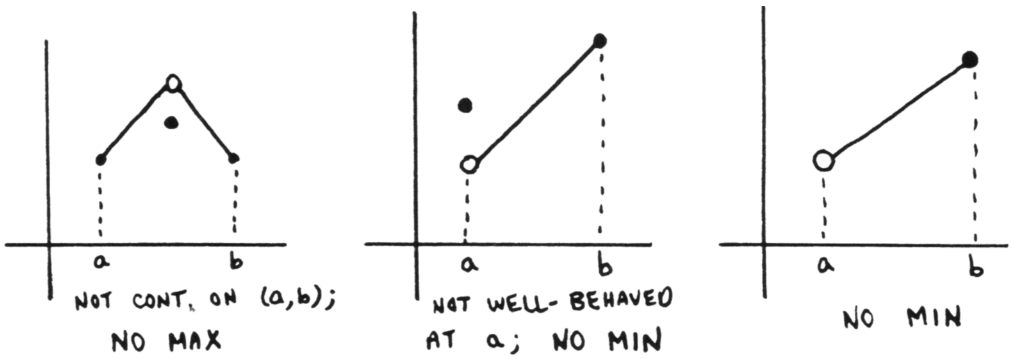
check that all the hypotheses are needed

To use the Max-Min Theorem, one must have a function f that is continuous on a closed interval $[a, b]$. That is, f must be defined on $[a, b]$, continuous on the open interval (a, b), and well-behaved at the endpoints. Take away any of these conditions, and extreme values are no longer guaranteed.

The first sketch below illustrates that continuity on (a, b) is needed.

The second sketch illustrates that the function must be well-behaved at the endpoints.

The third sketch illustrates that the function must be defined on a closed interval.



EXERCISE 3

♣ 1. Sketch the graph of a function that is NOT continuous on $[a, b]$, attains a minimum on $[a, b]$, does not attain a maximum on $[a, b]$.

♣ 2. Sketch the graph of a function that is NOT continuous on $[a, b]$, attains a maximum on $[a, b]$, but not a minimum.

♣ 3. Sketch the graph of a function that is NOT continuous on $[a, b]$, and attains both a maximum and minimum on $[a, b]$.

♣ 4. Sketch the graph of a function that is NOT continuous on $[a, b]$, and does not attain a maximum or minimum on $[a, b]$.

♣ 5. If f is NOT continuous on $[a, b]$, can the Max-Min Theorem be used to reach any conclusion about extreme values of f on $[a, b]$?

EXERCISE 4

♣ 1. Suppose you are given a function f and a closed interval I, and it is known that f does NOT attain a maximum value on I. Is f continuous on I?

♣ 2. Suppose f is defined on $[a, b]$ and continuous on (a, b). It is known that f does NOT attain a maximum value on $[a, b]$. Make some conclusion about the behavior of f on $[a, b]$.

In the next two chapters, calculus tools are developed to help locate maximum and minimum values, when they exist.

more on implications

This section is concluded with some additional study of *implications*. Note that the form of the Max-Min Theorem is an implication:

IF f is continuous on a closed interval $[a, b]$,

THEN f must take on both a maximum and minimum value on I.

the 'contrapositive' of an implication

The *contrapositive* of the implication

$$\text{If } A\text{, then } B$$

is another implication:

$$\text{If (not } B\text{), then (not } A\text{)}$$

EXAMPLE

finding contrapositives

The contrapositive of the true implication

$$x = 1 \implies x^2 = 1$$

is:

$$x^2 \neq 1 \implies x \neq 1$$

EXAMPLE

finding contrapositives

The contrapositive of the true implication

$$\text{If } f \text{ is continuous on } [a,b]\text{, then } f \text{ attains a maximum value on } [a,b]$$

is:

If f does not attain a maximum value on $[a,b]$, then f is not continuous on $[a,b]$

relationship between an implication and its contrapositive

Is there any nice relationship between an implication and its contrapositive? Where does intuition lead you? Roughly, a true sentence '*If A, then B*' says that whenever A is true, B must also be true. So if B isn't true, then A can't be true; because if A WERE true, B would have to be true. This is the intuition behind the result:

An implication is equivalent to its contrapositive.

That is:

$$\text{If } A\text{, then } B \iff \text{If (not } B\text{), then (not } A\text{)}$$

In alternate notation:

$$A \Rightarrow B \iff \text{not } B \Rightarrow \text{not } A$$

The proof is easy: just show that both sentences have precisely the same truth values, regardless of the truth values of A and B!

not B	not A	A	B	A ⇒ B	NOT B ⇒ NOT A
F	F	T	T	T	T
T	F	T	F	F	F
F	T	F	T	T	T
T	T	F	F	T	T

IDENTICAL !!

EXERCISE 5

Determine if the following implications are true or false. Then, find their contrapositives.

♣ 1. If $x \in [1, 2]$, then $x > 0$

♣ 2. If $x \in [0, 1)$, then $x > 0$

♣ 3. $x \in [0, 1) \implies x \ge 0$

♣ 4. If f is continuous on $[a, b]$, then f attains a minimum value on $[a, b]$.

♣ 5. Suppose that $a < b$, and D is a number between $f(a)$ and $f(b)$. Investigate this implication concerning f:

If f is continuous on $[a, b]$, then there exists a number $c \in [a, b]$ with $f(c) = D$.

QUICK QUIZ
sample questions

1. Let f be defined on an interval I containing c. Give a precise definition of the sentence, 'the number $f(c)$ is a maximum of f on I'.
2. Sketch the graph of a function f that is defined on $I := [1, 3]$, has a minimum value on I, but has no maximum value on I.
3. Sketch the graph of a function f that is continuous on (a, b) but attains NO maximum or minimum value on (a, b).
4. Give a precise statement of the Max-Min Theorem.
5. What is the contrapositive of $A \implies B$? What is the relationship between an implication and its contrapositive?

KEYWORDS
for this section

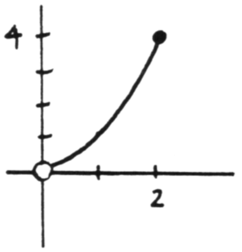
Extreme values for a function on an interval, extreme values may or may not exist, extreme values versus extreme points, the Max-Min Theorem, the contrapositive of an implication.

END-OF-SECTION EXERCISES

♣ Sketch the graph of each function f on the given interval I.

♣ Find the maximum and minimum value of f on I, if they exist.

♣ List all maximum points and minimum points (if any).



Be sure to answer using complete mathematical sentences. Here's a sample problem.

SAMPLE: $f(x) = x^2$, $I = (0, 2]$

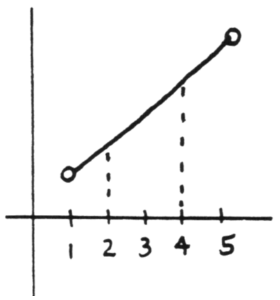
SOLUTION: The graph is shown at left. The maximum value of f on I is 4; there is no minimum value. The only maximum point is $(2, 4)$.

1. $f(x) = x^2$, $I = [0, 2)$
2. $f(x) = x^2$, $I = (0, 2)$
3. $f(x) = 4$, $I = \mathbb{R}$
4. $f(x) = -2$, $I = (0, \infty)$
5. $f(x) = (x-2)^2 + 1$, $I = (1, 3)$
6. $f(x) = (x-2)^2 + 1$, $I = [1, 3)$
7. $f(x) = |2x + 1|$, $I = (-1, 2]$
8. $f(x) = |2x + 1|$, $I = [-\frac{3}{4}, 0)$

♣ Determine if the following implications are true or false.

♣ If an implication is false, give a counterexample.

♣ Then, find the contrapositive of the implication.



Here's a sample problem:

SAMPLE: If f is continuous on $(1, 5)$, then f attains a maximum value on $(2, 4)$

SOLUTION: FALSE. Let f be the function graphed at left. Then the hypothesis 'f is continuous on $(1, 5)$' is TRUE, but the conclusion 'f attains a maximum value on $(2, 4)$' is FALSE.

The contrapositive is: If f does not attain a maximum value on $(2, 4)$, then f is not continuous on $(1, 5)$. (The contrapositive is of course also false.)

9. If f is continuous on $[a, b]$, then f attains a maximum value on $[a, b]$
10. If f does not attain a maximum value on $[a, b]$, then f is not continuous on $[a, b]$
11. If f is continuous on $(a, b]$, then f attains a maximum value on $(a, b]$
12. If f is continuous on $[a, b)$, then f attains a minimum value on $[a, b)$
13. If f is continuous on $(0, 5)$, then f attains both a maximum and minimum value on $[1, 2]$
14. If f is continuous on $(-5, -1)$, then f attains both a maximum and minimum value on $(-4, -2)$
15. If f is continuous on $\mathbb{R}$, then f attains a maximum value on $\mathbb{R}$; that is, there exists $c \in \mathbb{R}$ such that:
$$f(x) \le f(c) \quad \forall \;\; x \in \mathbb{R}$$
16. If f is continuous on $\mathbb{R}$, then f attains a minimum value on $\mathbb{R}$; that is, there exists $c \in \mathbb{R}$ such that:
$$f(c) \le f(x) \quad \forall \;\; x \in \mathbb{R}$$