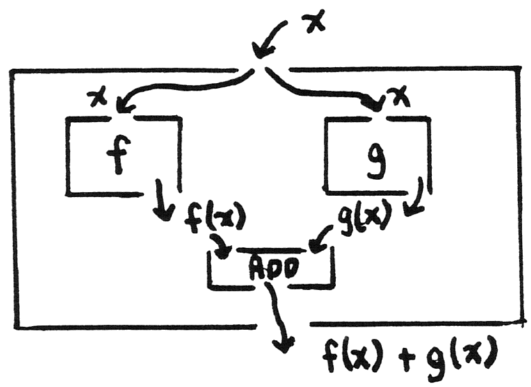
2.3 Composite Functions

using 'old' functions to define 'new' functions

There are many ways that functions can be 'combined' to form new functions. For example, the sketch below illustrates how functions f and g can be combined to form a 'sum' function.



After a review of some set operations, several very simple combinations of functions are discussed. Then, a very important type of combination—composition of functions—is investigated. A good understanding of function composition is necessary to understand the Chain Rule in Chapter 4.

Why are set operations being reviewed?

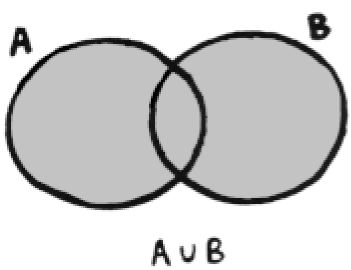
Every time that functions are combined to get new functions, one must determine how the domain of the new function is obtained from the domains of the old functions. Since the domain of a function is a *set*, finding this 'new' domain involves combining sets in various ways. The examples in this section will be greatly simplified if we first review some operations that have to do with combining sets.

set operation;

$A \cup B$,
A union B

We have seen that the *union of sets A and B* is defined by:

$$A \cup B := \{x \,|\, x \in A \text{ or } x \in B\}$$



This sentence is read as, *the set 'A union B' is defined as the set of all x with the property that x is in A, or x is in B.* The word 'or' is being used in a mathematical sense. This definition tells how an element gets in the 'new' set $A \cup B$; it must be an element for which the mathematical sentence

$$x \in A \text{ or } x \in B$$

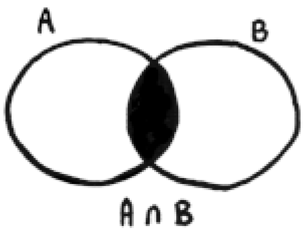
is true. When is this sentence true? By definition of the mathematical word 'or', it is true if $x \in A$ is true, or if $x \in B$ is true, or if both $x \in A$ and $x \in B$ are true.

So, this FACT is telling us that to form $A \cup B$ from sets A and B, we merely put in everything from A (the things that make $x \in A$ true) and everything from B (the things that make $x \in B$ true). Note that FACTS CAN TELL YOU WHAT TO DO.

set operation;
$A \cap B$,
A intersect B

There is another useful set operation called *set intersection*, defined as follows: Let A and B be sets. Define a new set $A \cap B$ (read as 'A intersect B') by:

$$A \cap B := \{x \,|\, x \in A \text{ and } x \in B\}$$



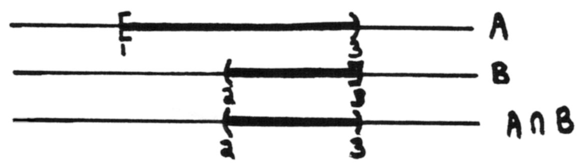
This definition says that for an element x to be in the set $A \cap B$, x must make the sentence

$$x \in A \quad \text{and } x \in B$$

true. The word 'and' is being used in the mathematical sense. When is this sentence true? By definition of the mathematical word 'and', it is true only when *both* $x \in A$ and $x \in B$ are true. So, the only elements that get into $A \cap B$ are those that are in *both* of the sets A and B.

For example, if $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, then $A \cap B = \{3, 4\}$.

As a second example, if $A = [1, 3)$ and $B = (2, 3]$ then $A \cap B = (2, 3)$. You should be able to tell, from context, that the parentheses and brackets denote *interval notation* here.



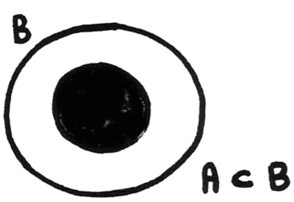
comparing sets;
$A \subset B$,
A is a subset of B
or
A is contained in B

Sometimes it is useful to know that one set is *contained* in another set. The *subset* symbol '$\subset$' is used in this situation. For sets A and B, the sentence $A \subset B$ means that everything in A is also in B. The sentence $A \subset B$ is read as '*A is a subset of B*' or '*A is contained in B*'.

EXAMPLES
using subset notation correctly

For example, if $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$ then the sentence $A \subset B$ is true. The sentence can also be written as $\{1, 2, 3\} \subset \{1, 2, 3, 4\}$.

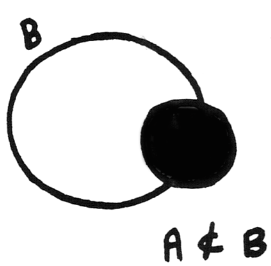
If A is *any* set, then the sentence $A \subset A$ is true. This is because any element of A (the set to the left of the $\subset$ symbol) is an element of A (the set to the right of the $\subset$ symbol).



The sentence $\{0, 1\} \subset [0, 1]$ is true; this is because $0 \in [0, 1]$ and $1 \in [0, 1]$. Note that the $\subset$ symbol is used to compare *sets*, so the symbols to the left and right of $\subset$ must be *sets*. However, the $\in$ symbol must have an *element* on the left and a *set* on the right.

If $A = \{0, 1, 2\}$ and $B = \{1, 2\}$, then the sentence $A \subset B$ is NOT true. This is because 0 is an element of A, but 0 is not an element of B.

The sentence $\{0, 1\} \subset (0, 1]$ is NOT true. This is because $0 \notin (0, 1]$.



Suppose that A is a set with element a, and B is a set with element b. Then all the following sentences are true, and illustrate the correct use of the symbols $\subset$ and $\in$:

- $a \in A$
- $\{a\} \subset A$
- $a \in A \cup B$
- $\{a, b\} \subset A \cup B$

EXERCISE 1

practice with set operations

♣ 1. True or False:

a) If A and B are sets, then $A \cup B$ is a set.

b) If C and D are sets, then $C \cap D$ is a set.

c) If A and B are sets, then $A \subset B$ is a set.

d) $\{0, 2, 4\} \subset \mathbb{Z}$

e) $\mathbb{Q} \subset \mathbb{Z}$

f) $\mathbb{Z} \subset \mathbb{Q}$

g) $\{0\} \in \{0, 1, 2\}$ (Be careful!)

♣ 2. Let $A = \{0, 1, 2, 3, 4, 5\}$ and $B = \{x \in \mathbb{Z} \,|\, x \geq 3\}$. Find the following sets:

a) $A \cup B$

b) $A \cap B$

c) $\mathbb{R} \cap A$

d) $\mathbb{Z} \cap B$

e) $\mathbb{Q} \cap (A \cup B)$

f) $(0, 6] \cap A$

♣ 3. For the sets A and B defined above, are the following sentences true or false?

a) $A \subset B$; Why or why not?

b) $B \subset A$; Why or why not?

♣ 4. True or False: For all sets C and D, the mathematical sentence

$$C \subset D \text{ or } D \subset C$$

is true.

♣ 5. True or False: For all sets C and D, the mathematical sentence

$$C \subset C \cup D$$

is true.

functions can take all kinds of inputs and give all kinds of outputs

In general, functions can take all kinds of inputs, and give all kinds of outputs. For example, one could define a function f that takes a NAME as an input, and gives the first letter of the name as the output. For this function, then, $f(Carol) = C$ and $f(Robert) = R$.

In more advanced mathematics courses, many important functions take *other* functions as inputs, and give functions as outputs!

ASSUMPTION IN THIS TEXT

all functions are assumed to be of this type: $f\colon \mathcal{D}(f) \to \mathbb{R}$, with $\mathcal{D}(f) \subset \mathbb{R}$

In this course, the functions that we deal with primarily are those that take a single input (a real number), and give a single output (a real number). For ease of notation, then, *henceforward in this text, unless otherwise stated, functions are ASSUMED to be functions of one variable, where both the input and output are real numbers.* That is, all functions are assumed to be of the form

$$f\colon \mathcal{D}(f) \to \mathbb{R} \ ,$$

where $\mathcal{D}(f) \subset \mathbb{R}$.

Now we are in a position to begin talking precisely about combining functions to get new functions.

the sum function
$(f+g)(x) := f(x)+g(x)$

Consider functions f and g. Define a new function, with name $f+g$, by the following rule:

$$(f+g)(x) := f(x) + g(x)$$

What does this function $f+g$ DO? Answer:

- it takes an input x
- it lets f act on x to get $f(x)$
- it lets g act on x to get $g(x)$
- it gives, as its output, the sum $f(x) + g(x)$

What is the domain of this new function $f+g$? In order for f to know how to act on x, we must have $x \in \mathcal{D}(f)$. In order for g to know how to act on x, we must also have $x \in \mathcal{D}(g)$. *Any* two real numbers $f(x)$ and $g(x)$ can be added. Thus:

$$\begin{aligned}\mathcal{D}(f+g) &= \{x \,|\, x \in \mathcal{D}(f) \text{ and } x \in \mathcal{D}(g)\} \\ &= \mathcal{D}(f) \cap \mathcal{D}(g)\end{aligned}$$

EXERCISE 2

♣ 1. Let $f(x) = x^2$ and $g(x) = \sqrt{x}$. Find the new function $f+g$. What is the domain of $f+g$? Be sure to write complete mathematical sentences.

♣ 2. Given arbitrary functions f and g, define a new function $f-g$ in the natural way. What is the domain of $f-g$? Be sure to write complete mathematical sentences.

♣ 3. Given arbitrary functions f and g, what should a function with the name fg do? Write down a precise definition. What is the domain of fg? Be sure to write complete mathematical sentences.

EXERCISE 3

Let f be any function, and let $k \in \mathbb{R}$. Define a new function kf by the rule:

$$(kf)(x) := k \cdot f(x)$$

♣ 1. What does the dot '$\cdot$' mean in the definition above? You should be able to figure this out from context.

♣ 2. In words, what does the function kf do?

♣ 3. What is the domain of the function kf? Write a complete mathematical sentence.

♣ 4. If $f(x) = x^3$ and $k = 4$, what is kf?

quotient function
$(\frac{f}{g})(x) := \frac{f(x)}{g(x)}$

As a second example, consider functions f and g. Define a new function, with name $\frac{f}{g}$, by the rule:

$$(\frac{f}{g})(x) := \frac{f(x)}{g(x)}$$

What is the domain of this new function $\frac{f}{g}$? There are three things to worry about. Firstly, f must know how to act on x in order to get $f(x)$, so we must have $x \in \mathcal{D}(f)$. Similarly, we must have $x \in \mathcal{D}(g)$. But there is an additional requirement; since division by zero is not allowed, we must also have $g(x) \neq 0$. Thus:

$$\mathcal{D}(\frac{f}{g}) = \{x \,|\, x \in \mathcal{D}(f) \text{ and } x \in \mathcal{D}(g) \text{ and } g(x) \neq 0\}$$

★★
associativity of the logical 'and'

An important point is being glossed over here. The logical 'and' is associative; that is,

$$(A \wedge B) \wedge C \iff A \wedge (B \wedge C) \ ,$$

as the truth table below illustrates. This allows us to say, without ambiguity, 'A and B and C' (without parentheses). Since intuition leads to this result anyway, students should believe the preceding argument without any digression.

A	B	C	$A \wedge B$	$(A \wedge B) \wedge C$	$B \wedge C$	$A \wedge (B \wedge C)$
T	T	T	T	**T**	T	**T**
T	T	F	T	**F**	F	**F**
T	F	T	F	**F**	F	**F**
T	F	F	F	**F**	F	**F**
F	T	T	F	**F**	T	**F**
F	T	F	F	**F**	F	**F**
F	F	T	F	**F**	F	**F**
F	F	F	F	**F**	F	**F**
				↑ Same!		↑ Same!

EXERCISE 4

♣ 1. Define a function named $\sqrt{f}$ by the rule:

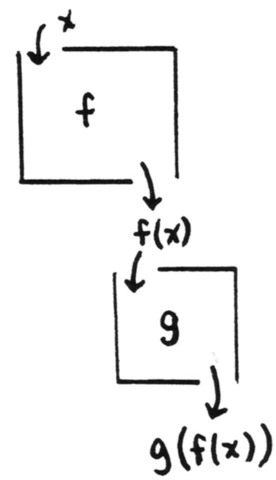
$$(\sqrt{f})(x) := \sqrt{f(x)}$$

What is the domain of $\sqrt{f}$? Be sure to write a complete mathematical sentence.

♣ 2. If $f(x) = x^3$, what is $\mathcal{D}(\sqrt{f})$?

♣ 3. If $g(x) = -x^2$, what is $\mathcal{D}(\sqrt{g})$?

composite functions



We next talk about a *very important* way of 'combining' functions: function composition.

Consider the 'combination' of functions illustrated in the diagram at left. Here, an input x is dropped into the f box, giving the output $f(x)$. Then, this output $f(x)$ is dropped into the g box, giving the output $g\big(f(x)\big)$. This type of combination of functions—characterized by one box acting in series with another box—is called *function composition.*

More precisely, consider functions f and g. A new function, named $g \circ f$ (read as 'g circle f' or 'g composed with f') is defined by the rule:

$$(g \circ f)(x) := g\big(f(x)\big)$$

This notation can be a little tricky: although the 'g' *appears* first in the name $g \circ f$, it doesn't ACT first! In the function $g \circ f$, the right-most function f acts first (it is 'closest' to x), then g acts. Analyze the definition again:

$$(g \circ f)(x) := \overbrace{g\Big(\quad \underbrace{f(x)}_{f \text{ acts on } x \text{ first}} \quad \Big)}^{g \text{ acts last}}$$

EXERCISE 5
practice with composition

♣ 1. Write down a precise definition of the function $f \circ g$. Draw a series of boxes that describes this function. Which function acts first?

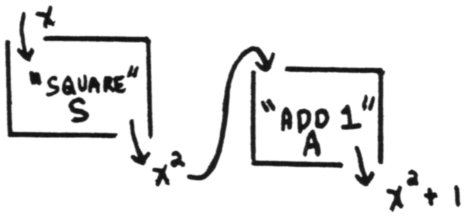
♣ 2. More than two functions can be composed. For example, one can define a new function $f \circ g \circ h$ by the rule:

$$(f \circ g \circ h)(x) := f(g(h(x)))$$

What function acts first? Second? Third? Draw a series of boxes that describes $f \circ g \circ h$.

EXAMPLE
viewing a function as a composition

Consider the function f defined by the rule $f(x) = x^2 + 1$. This function takes an input, squares it, and then adds 1. Thus, it can be viewed as a composition:



Define functions S (for 'square') and A (for 'add') by $S(x) = x^2$ and $A(x) = x + 1$. Then:

$$(A \circ S)(x) = A(S(x)) = A(x^2) = x^2 + 1$$

Thus, the function f has been viewed as a composition $A \circ S$.

EXERCISE 6

Consider the function f defined by the rule $f(x) = (x + 1)^2$.

♣ 1. Describe, in words, what f does to a typical input x.

♣ 2. Draw a series of 'boxes' that describe what f does.

♣ 3. As in the previous example, view f as a composition of functions.

domain of the function $g \circ f$

Now, return to the function $g \circ f$ defined by the rule:

$$(g \circ f)(x) := g\big(f(x)\big)$$

What is the domain of this new function $g \circ f$? There are two things to worry about. Firstly, f must know how to act on x, so we must have $x \in \mathcal{D}(f)$. Secondly, g must know how to act on $f(x)$. That is, outputs from f are only acceptable IF they happen to be in the domain of g. To say this precisely:

$$\mathcal{D}(g \circ f) = \{x \mid x \in \mathcal{D}(f) \text{ and } f(x) \in \mathcal{D}(g)\}$$

EXAMPLE
finding $\mathcal{D}(g \circ f)$

Let f and g be defined by $f(x) = -x^3$ and $g(x) = \sqrt{x}$. What is $\mathcal{D}(g \circ f)$? First,

$$\begin{aligned}(g \circ f)(x) &:= g\big(f(x)\big) \\ &= g(-x^3) \\ &= \sqrt{-x^3}\end{aligned}$$

By the domain convention, then:

$$\mathcal{D}(g \circ f) = \{x \,|\, -x^3 \ge 0\} = (-\infty, 0]$$

Although $\mathcal{D}(f) = \mathbb{R}$, not all these inputs are allowed for the new function $g \circ f$. It is also required that $f(x)$ (the input to g) be nonnegative; and this happens only when $x \le 0$.

A slightly different technique is illustrated to find the domain of the 'reverse' composite, $\mathcal{D}(f \circ g)$:

$$\begin{aligned}\mathcal{D}(f \circ g) &= \{x \,|\, x \in \mathcal{D}(g) \text{ and } g(x) \in \mathcal{D}(f)\} \\ &= \{x \,|\, x \ge 0 \text{ and } \sqrt{x} \in \mathbb{R}\} \\ &= \{x \,|\, x \ge 0\} \\ &= [0, \infty)\end{aligned}$$

Be sure that you understand every step in this mathematical sentence!

EXERCISE 7

♣ In the example above, the domain of $g \circ f$ was found to be $(-\infty, 0]$. Find this result a different way, by completing the following mathematical sentence:

$$\begin{aligned}\mathcal{D}(g \circ f) &= \{x \,|\, x \in \mathcal{D}(f) \text{ and } f(x) \in \mathcal{D}(g)\} \\ &= \{x \,|\, x \in \text{???} \text{ and } \text{?????????}\} \\ &= \text{?????}\end{aligned}$$

EXERCISE 8

Define functions f and g by $f(x) = |x|$ and $g(x) = \frac{1}{x}$.

♣ 1. Write a formula for $f \circ g$, and find its domain.

♣ 2. Write a formula for $g \circ f$, and find its domain.

♣ 3. Are $f \circ g$ and $g \circ f$ the same functions in this case? (That is, do they have the same domains, and use the same rule to obtain their outputs?)

♣ 4. Is it *always* true that $f \circ g = g \circ f$? If not, give an example where $f \circ g \ne g \circ f$.

When forming the composition $g \circ f$ from functions f and g, the nicest possible situation is when g knows how to act on ALL the outputs $f(x)$. We need a name for this set of outputs from f, and this is the next topic of discussion.

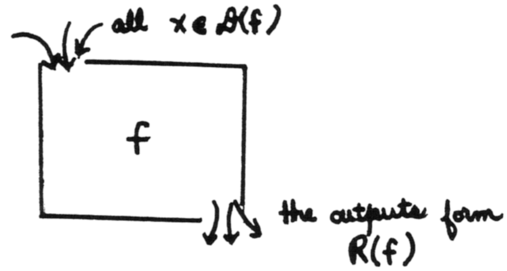
DEFINITION
the range of a function f, $\mathcal{R}(f)$

Let f be a function with domain $\mathcal{D}(f)$. Then, the *range of f*, denoted by $\mathcal{R}(f)$, is the set of all outputs obtained from f as x takes on all possible input values. Precisely:

$$\mathcal{R}(f) := \{f(x) \,|\, x \in \mathcal{D}(f)\}$$

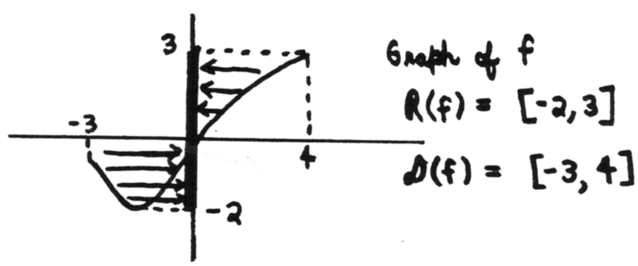
black box interpretation of the range

Here's the 'black box' interpretation of the range: drop *all possible* inputs into the top of the box. Gather together all the outputs that come out of the box. This set forms the range.



finding the range from a graph

If the graph of a function is available, then it is easy to determine the range: just imagine 'collapsing' the graph into the y-axis. The set of all y-values that are taken on forms the range of the function.



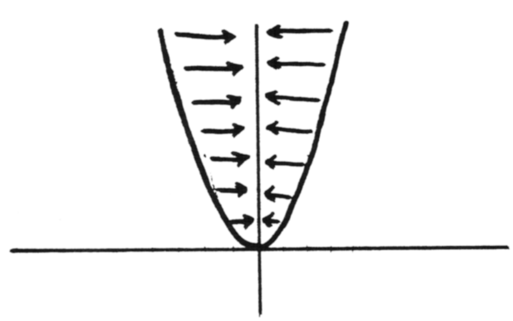
EXAMPLE

finding $\mathcal{R}(f)$

Let f be the function defined by the rule $f(x) = x^2$. By the domain convention, $\mathcal{D}(f) = \mathbb{R}$. What is the range?

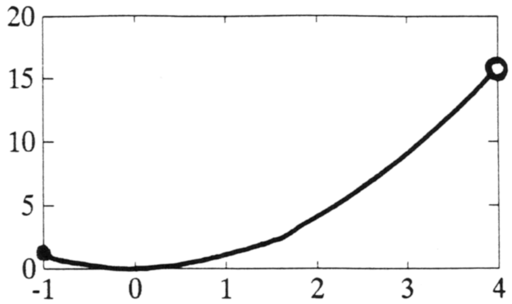
$$\begin{aligned}\mathcal{R}(f) &:= \{f(x) \mid x \in \mathcal{D}(f)\} \\ &= \{x^2 \mid x \in \mathbb{R}\} \\ &= [0, \infty)\end{aligned}$$

The question is also easily answered by studying the graph of f. If the graph is 'collapsed' into the y-axis, one obtains the interval $[0, \infty)$.



EXAMPLE
finding $\mathcal{R}(f)$

Now consider $f\colon [-1,4) \to \mathbb{R}$, $f(x) = x^2$. The graph of f is shown below:



As f runs through all the elements in its domain, the only outputs obtained are those in the set $[0,16)$. Thus:

$$\mathcal{R}(f) = [0,16)$$

EXAMPLE
finding the range of a constant function

Consider the constant function given by the rule $g(x) = 16$. By the domain convention, $\mathcal{D}(g) = \mathbb{R}$. But as g acts on all possible inputs, the only output obtained is 16. Thus, $\mathcal{R}(g) = \{16\}$. (♣ Why is it INCORRECT to say $\mathcal{R}(g) = 16$?)

EXERCISE 9
finding the range

Find the domains and ranges of the following functions. Be sure to write complete mathematical sentences.

♣ 1. $f_1(x) = \sqrt{x} + 1$

♣ 2. $f_2(x) = \sqrt{x+1}$

♣ 3. $g\colon [0,4] \to \mathbb{R}$ given by $g(x) = \sqrt{x}$

♣ 4. $h\colon \{0\} \cup (1,3] \to \mathbb{R}$ given by $h(x) = \sqrt{x}$

QUICK QUIZ
sample questions

1. Let A and B be sets. Write a precise definition of $A \cap B$. If $A = [1,3)$ and $B = \{1,2,3\}$, what is $A \cap B$?
2. TRUE or FALSE:
   - $[1,3] \subset \{1,3\}$
   - $\{1,3\} \subset [1,3]$
   - For all sets A and B, $A \cap B \subset A$.
3. Let f and g be functions. Give a precise definition of the new function $f+g$. What is the domain of $f+g$? Be sure to write a complete mathematical sentence.
4. Define functions a and b so that the function $f(x) = 2x - 1$ can be written as a composition, $f = a \circ b$.
5. What is the range of the function $f\colon \mathbb{Z} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 1 & \text{for } n > 1 \\ -1 & \text{for } n \le 1 \end{cases} \ ?$$

KEYWORDS *for this section*	*Set union ($A \cup B$), set intersection ($A \cap B$), subset notation ($A \subset B$), assumption about functions in this text, combining functions to form new functions, determining the domain of the 'new' function, composite functions ($f \circ g$), domain of $f \circ g$, range of a function f, notation $\mathcal{R}(f)$.*

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN). The context will determine if a variable is a number, a function, or a set.

♣ For any *sentence*, state whether it is TRUE (T), FALSE (F), or CONDITIONAL (C).

1. $A \cup B$
2. $A \subset A \cup B$
3. $A \subset B$
4. $\mathcal{R}(f)$
5. $\mathcal{R}(f) = \mathbb{R}$
6. $\{x \mid x \in \mathcal{D}(f) \text{ and } f(x) \in \mathcal{D}(g)\}$
7. $x \in \mathcal{D}(f)$ and $f(x) \in \mathcal{D}(g)$
8. $(f + g)(x) := f(x) + g(x)$
9. $\{a\} \in \{a, b\}$
10. $\{a\} \subset \{a, b\}$

Find the range of each of the following functions. (You graphed these functions in §2.2, End-Of-Section Exercises, 11–14.)

11. $f\colon [-1, 1] \to \mathbb{R},\ \ f(x) = (x + 1)^3$
12. $g\colon (-\infty, 4] \to \mathbb{R},\ \ g(t) = 3|t - 2| - 1$
13. $h\colon \{1, 4, 9, 16, 25\} \to \mathbb{R},\ \ h(t) = \sqrt{t}$
14. $f\colon \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\} \to \mathbb{R},\ \ f(x) = \frac{1}{x}$