6. AVERAGE

introduction

A teacher reports an average grade on a test. You read about the average number of calories burned per hour for your favorite exercise. What do these figures mean? The purpose of this section is to discuss the concept, the computation, and some important properties of averaging.

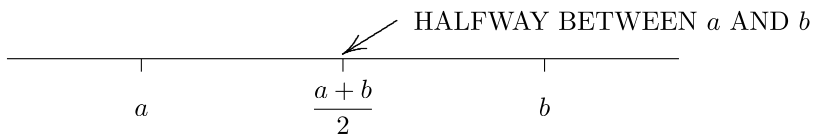
averaging two numbers

To *average two numbers* means to add the numbers together, and then divide by 2. Thus,

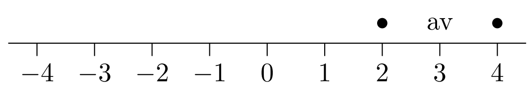
$$\text{the average of } a \text{ and } b \text{ is } \frac{a+b}{2} .$$

averaging gives the midpoint

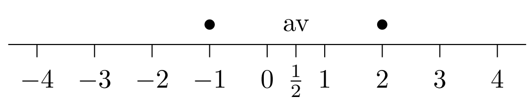
Averaging two different numbers always gives the number exactly halfway between, as illustrated below.



For example, averaging 2 and 4 gives $\dfrac{2+4}{2} = \dfrac{6}{2} = 3$:



Averaging -1 and 2 gives $\dfrac{-1+2}{2} = \dfrac{1}{2}$:



computing the average of two numbers

In the web exercises for this section you will practice computing averages of two numbers, where the numbers can be $-10, -9, \ldots, -1, 0, 1, \ldots, 8, 9, 10$. You must be able to do these exercises *without* a calculator! This is good practice with mental arithmetic, and will reinforce your skills with addition of signed numbers. There are several key ideas you will want to keep in mind:

- If the two numbers being averaged are close to each other, just visualize the number line and picture the number that is exactly halfway between. For example, the average of 3 and 5 is 4. The average of 7 and -7 is 0. The average of -1 and -2 is -1.5.
- If the numbers being averaged are far enough apart that you can't easily decide which number is halfway between, then do the arithmetic. Add the two numbers and divide by 2. For example, the average of 5 and -7 is $\frac{5+(-7)}{2} = \frac{-2}{2} = -1$.
- On the web exercises, you must report your answers in decimal form. For example, $\frac{5}{2}$ must be reported as 2.5. If you're a bit rusty with fractions and decimals, don't worry—they'll be reviewed in future sections. For now, you just need to be able to convert fractions that have a denominator of 2 to decimal form, and this idea is discussed in the following paragraph.

converting fractions with a denominator of 2 *to decimal form*

To convert $\frac{15}{2}$ to decimal form, go through this thought process: How many times does 2 go into 15? It goes in 7 times with 1 left over. The answer is 7.5.

To convert $\frac{-19}{2}$ to decimal form, go through this thought process: Firstly, the answer will be negative. How many times does 2 go into 19? Well, it goes in 9 times with 1 left over. The answer is -9.5.

The web exercises for section 6 give you unlimited practice converting fractions with a denominator of 2 into decimal form.

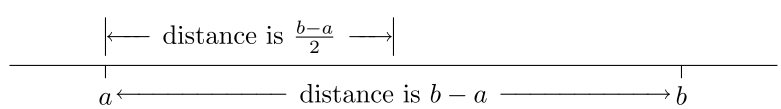
How do we know that $\frac{a+b}{2}$ really lies halfway between a and b?

Clearly, the formula $\frac{a+b}{2}$ gives *some* number; but how do we know that the number given by this formula is *really*, *always*, halfway between a and b? Although repeated trials (with lots of different numbers) is pretty convincing, it is of course impossible to check *every* pair of real numbers. Thus, mathematicians prefer to prove their point with an argument like the one shown below. (Remember that ★ material can be skipped without any loss of continuity.)

★
algebraic proof for more experienced readers; the average of two numbers gives a number that is exactly halfway between

Let a and b be different real numbers; rename, if necessary, so that $a < b$. The distance between a and b is $b - a$, and half this distance is $\frac{b-a}{2}$. Then, the midpoint of a and b is:

$$a + \frac{b-a}{2} = \frac{2a}{2} + \frac{b-a}{2} = \frac{2a+b-a}{2} = \frac{a+b}{2}\ .$$



EXERCISES

1. a. Find the average of 2 and 6.
   b. Find the number exactly halfway between 2 and 6 on a number line. Compare with (a).
2. The numbers 0.13 and 0.14 are very close to each other. Find a number halfway between them. (Use a calculator, if necessary.)
3. What happens if you average two numbers that are the same?

averaging more than two numbers

To average N numbers, add them up and divide by N.

subscript notation

A convenient way to talk about N numbers is to use *subscript notation*. A *subscript* is a number or letter that is written slightly below another character. For example, when you look at x_3 (read as 'x sub three'), 3 is a subscript. When you look at y_b (read as 'y sub b'), b is a subscript.

In subscript notation, we let x_1 (read as 'ex sub one') denote the first number, x_2 (read as 'ex sub two') denote the second number, and so on.

So, we can let $x_1, x_2, \ldots, x_N$ denote N numbers. Then, the average of these N numbers is:

$$\frac{x_1 + x_2 + \cdots + x_N}{N}.$$

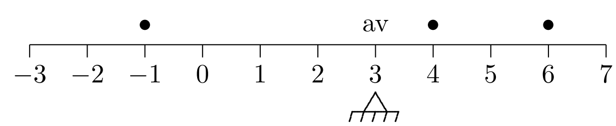
EXERCISES

4. Write the formula for the average of four numbers. Use subscript notation.
5. Write the formula for the average of M numbers. Use subscript notation.

more than two numbers

When more than two numbers are averaged, the concept of 'balancing point' becomes the central idea. To illustrate the idea, consider finding the average of three numbers: -1, 4, and 6.

Put equal-size pebbles at locations -1, 4 and 6 on the number line. If you think of the number line as a see-saw from a childhood playground, the support must be placed at the average, $\frac{-1+4+6}{3} = \frac{9}{3} = 3$, for perfect balance!



the average always lies between the greatest and least numbers

It is clear from the 'balancing point' interpretation of the average that the average of numbers always lies between the greatest number (the one farthest to the right) and the least number (the one farthest to the left).

EXERCISES

6. Suppose that fifteen positive numbers are being averaged. Each of the numbers being averaged is less than 50. What, if anything, can be said about the average?
7. Suppose that x is a number. If you average the numbers $x+2$ and $x-2$, what do you get?

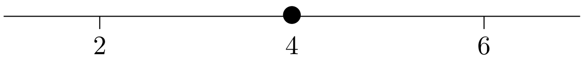
EXERCISES
web practice

8. Go to **http://fishcaro.crosswinds.net** and follow the links to the practice problems for section 6. Here you will practice converting a fraction with a denominator of 2 to decimal form; finding the averages of two numbers; and finding the average of three numbers. In the process, you'll get lots of practice with addition of signed numbers. For your convenience, there are also worksheets provided in this text on the following pages. Additional worksheets can be produced at the web site.

SOLUTIONS TO EXERCISES: AVERAGE

1a. The average of 2 and 6 is $\frac{2+6}{2} = \frac{8}{2} = 4$.

1b.



The answers to both (a) and (b) are the same, since the average of two numbers gives the number that is exactly halfway between.

2. The average of 0.13 and 0.14 is $\frac{0.13+0.14}{2} = \frac{0.27}{2} = 0.135$. The number 0.135 is exactly halfway between 0.13 and 0.14 .

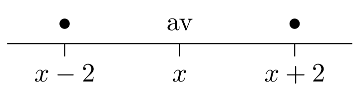
3. If you average two numbers that are the same, then you get the same number. That is, $\frac{x+x}{2} = \frac{2x}{2} = x$.

4. $\frac{x_1 + x_2 + x_3 + x_4}{4}$

5. $\frac{x_1 + x + 2 + \cdots + x_M}{M}$

6. The average must be between 0 and 50 .

7. x lies exactly halfway between $x-2$ and $x+2$:



$\frac{40+41+42+43+44}{5}$