35. SOLVING ABSOLUTE VALUE EQUATIONS

solving equations involving absolute value

This section presents the tool needed to solve absolute value equations like these:

$$|x| = 5$$
$$|2 - 3x| = 7$$
$$5 - 2|3 - 4x| = -7$$

Each of these equations has only a single set of absolute value symbols, and has a *variable* inside the absolute value. Solving sentences like these is easy, if you remember the critical fact that

absolute value gives distance from 0.

Keep this in mind as you read the following theorem:

THEOREM

tool for solving absolute value equations

Let $x \in \mathbb{R}$, and let $k \geq 0$. Then,

$$|x| = k \iff x = \pm k\ .$$

$|x| = k$ is an entire class of sentences

Recall first that normal mathematical conventions dictate that '$|x| = k$' represents an entire class of sentences, including $|x| = 2$, $|x| = 5.7$, and $|x| = \frac{1}{3}$. The variable k changes from sentence to sentence, but is constant within a given sentence.

EXERCISES

1. a. Give three sentences of the form '$|x| = k$' where $k \geq 0$. Use examples different from those given above.
   b. Give three sentences of the form '$x = \pm k$' where $k \geq 0$.

translating the theorem:

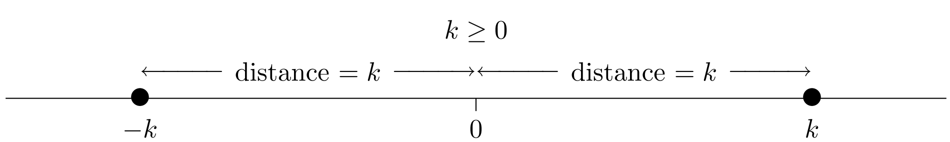
thought process for solving sentences like $|x| = k$

When you see a sentence of the form '$|x| = k$', here's what you should do:

- Check that k is a nonnegative number.
- The symbol $|x|$ represents the distance between x and 0.
- Thus, you want the numbers x, whose distance from 0 is k.

$$\overbrace{|x|}^{\text{want numbers } x, \text{ whose distance from } 0} \quad \overbrace{=}^{\text{is}} \quad \overbrace{k}^{k}$$

- You can walk from 0 in two directions: to the right, or to the left. Thus, there are two numbers whose distance from 0 is the nonnegative number k: k and $-k$.



- Thus, $|x| = k$ is equivalent to $x = \pm k$.

filling in some blanks to help your thought process

When you see a sentence like ‘ $|x| = 7$ ’, you thought process should be like filling in the following blanks:

We want the numbers _____ , whose _____ from _____ is _____ . Thus, we want _____ to be _____ or _____ .

The correctly-filled-in blanks are:

We want the numbers $\underline{x}$, whose <u>distance</u> from $\underline{0}$ is $\underline{7}$. Thus, we want $\underline{x}$ to be $\underline{7}$ or $\underline{-7}$.

EXERCISES

2. Fill in the blanks:
   a. When you look at the sentence ‘ $|x| = 5$ ’, you should think: We want the numbers _____ , whose _____ from _____ is _____ . Thus, we want _____ to be _____ or _____ .
   b. When you look at the sentence ‘ $|z| = \frac{1}{5}$ ’, you should think: We want the numbers _____ , whose _____ from _____ is _____ . Thus, we want _____ to be _____ or _____ .
   c. When you look at the sentence ‘ $|x| = k$ ’ (with $k \geq 0$), you should think: We want the numbers _____ , whose _____ from _____ is _____ . Thus, we want _____ to be _____ or _____ .
3. Give a sentence, not using absolute value symbols, that is equivalent to:
   a. $|x| = 3$
   b. $|t| = 4.2$
4. Give a sentence, using absolute value symbols, that is equivalent to:
   a. $x = \pm 7$
   b. $t = \frac{1}{3}$ or $t = -\frac{1}{3}$
5. Give a precise mathematical statement of the tool that says that a sentence like ‘ $|x| = 5$ ’ can be transformed to the equivalent sentence ‘ $x = \pm 5$ ’.
6. Is the sentence ‘ $|x| = -6$ ’ of the form described in the previous theorem? Why or why not?
7. Can the sentence ‘ $|x| - 5 = 7$ ’ be transformed to a sentence of the form described in the previous theorem? If so, what is the equivalent sentence?

EXAMPLE

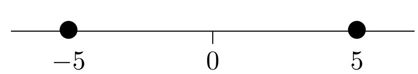
solving a sentence of the form $|x| = k$

Example: Solve: $|x| = 5$

Solution:

$$|x| = 5$$
$$x = \pm 5$$

The solution set is shaded below:



solving more complicated sentences of the form $|x| = k$ *;*

x can be ANYTHING!

The power of the tool

$$|x| = k \iff x = \pm k$$

goes way beyond solving simple sentences like '$|x| = 5$'. Since x can be *any* real number, you should think of x as merely representing *the stuff inside the absolute value symbols*. Thus, you could think of rewriting the tool as:

$$|\text{stuff}| = k \iff \text{stuff} = \pm k$$

Thus, we have all the following equivalences:

$$|\overbrace{2-3x}^{\text{stuff}}| = 7 \iff \overbrace{2-3x}^{\text{stuff}} = \pm 7$$

$$|\overbrace{5x-1}^{\text{stuff}}| = 8 \iff \overbrace{5x-1}^{\text{stuff}} = \pm 8$$

$$|\overbrace{x^2-3x+4}^{\text{stuff}}| = \frac{1}{5} \iff \overbrace{x^2-3x+4}^{\text{stuff}} = \pm\frac{1}{5}$$

and so on!

EXERCISES

8. For each of the following, write an equivalent sentence that does not use absolute value symbols. Do *not* solve the resulting sentences.
   a. $|1 + 2x| = 3$
   b. $|7x - \frac{1}{2}| = 5$
   c. $|x^2 - 8| = 0.4$

EXAMPLE

Example: Solve: $|2 - 3x| = 7$

Solution: Be sure to write a nice, clean list of equivalent sentences.

$\lvert 2 - 3x\rvert = 7$	original sentence
$2 - 3x = \pm 7$	$\lvert x\rvert = k \iff x = \pm k$
$-3x = \pm 7 - 2$	subtract 2 from both sides
$x = \frac{\pm 7 - 2}{-3}$	divide both sides by -3
$x = \frac{7-2}{-3}$ or $x = \frac{-7-2}{-3}$	$x = \pm k \iff (x = k \text{ or } x = -k)$
$x = -\frac{5}{3}$ or $x = 3$	arithmetic

Checking gives:

$$|2-3(-\frac{5}{3})| \stackrel{?}{=} 7$$
$$|2+5| \stackrel{?}{=} 7$$
$$7 = 7$$

$$|2-3(3)| \stackrel{?}{=} 7$$
$$|-7| \stackrel{?}{=} 7$$
$$7 = 7$$

EXAMPLE

an alternate approach to the previous example

Example: Here, the previous example is repeated, except this time without using the ' $\pm$ ' notation. Use whichever approach is more comfortable for you.

Solve: $|2-3x| = 7$

Solution:

$$|2-3x| = 7$$
$$2-3x = 7 \text{ or } 2-3x = -7$$
$$-3x = 5 \text{ or } -3x = -9$$
$$x = -\frac{5}{3} \text{ or } x = 3$$

an alternate form for the check

Here is an alternate way to write down the check, which is a bit more compact. This method works nicely when the the original equation has a constant on one side. Notice that the value of x is substituted into the side of the equation containing the variable, and it is shown to equal the desired constant.

$$|2-3(-\frac{5}{3})| = |2+5| = 7$$
$$|2-3(3)| = |-7| = 7$$

EXERCISES

9. Solve. Write a nice, clean list of equivalent sentences. Check your solutions.
   a. $|1+2x| = 3$
   b. $|7x-5| = 2$
   c. $|4-9x| = 1$

What happens if k is negative in the sentence '$|x| = k$'?

What about a sentence like '$|x| = -5$', where the absolute value is equal to a negative number? Notice that this situation is not covered in the previous theorem, since the sentence '$|x| = k$' is only addressed with $k \ge 0$.

Recall that $|x| \ge 0$ for all real numbers x. Thus, the sentence '$|x| = -5$' is never true: the left-hand side is always greater than or equal to 0, and the right-hand side is less than 0:

$$\overbrace{|x|}^{\ge 0} = \overbrace{-5}^{<0}$$

Even when x is 5 or -5, the sentence is false, as shown below:

x	substitution into '$\lvert x\rvert = -5$'	true or false?
5	$\lvert 5\rvert = -5$	false
-5	$\lvert -5\rvert = -5$	false

first step when analyzing $|x| = k$: check that $k \ge 0$

Whenever you're working with a sentence of the form '$|x| = k$', you must always check first that $k \ge 0$. If k is negative, you just stop and say that the sentence is always false. Here are some examples, which illustrate different ways that you can state your answer:

'$|x| = -3$' is always false.

'$|2x - 1| = -5$' is never true.

'$|3x - 5x^2 + 7| = -0.4$' has an empty solution set.

EXERCISES

10. Decide which of the following sentences are always false. Do NOT solve the sentences.
    a. $|x| = -9$
    b. $|x| = 0$
    c. $|3x - 5| = -4.7$
    d. $|1 - 4x| + 5 = 0$
    e. $-2|x^2 + 3x - 1| = 8$
    f. $|9x + 1| - 5 = -3$

EXAMPLE

putting a sentence in standard form first

Sometimes you need a few transformations to get an equivalent sentence in the form $|x| = k$, as the next example illustrates.

Solve: $5 - 2|3 - 4x| = -7$

$5 - 2\|3 - 4x\| = -7$	original sentence
$-2\|3 - 4x\| = -12$	subtract 5 from both sides
$\|3 - 4x\| = 6$	divide both sides by -2
$3 - 4x = 6$ or $3 - 4x = -6$	$\|x\| = k \iff x = \pm k$
$-4x = 3$ or $-4x = -9$	addition property of equality
$x = -\frac{3}{4}$ or $x = \frac{9}{4}$	multiplication property of equality

Checking:

$$5 - 2|3 - 4(-\frac{3}{4})| = 5 - 2|3 + 3| = 5 - 2|6| = 5 - 2(6) = 5 - 12 = -7$$

$$5 - 2|3 - 4(\frac{9}{4})| = 5 - 2|3 - 9| = 5 - 2|-6| = 5 - 2(6) = 5 - 12 = -7$$

EXERCISES

11. Solve and check each of the following equations. Be sure to write a nice, clean list of equivalent equations.
    a. $7 - 5|1 - 2x| = -3$
    b. $-3|2x - 1| - 5 = -4$
    c. $2|3x - 5| - 1 = 7$

EXERCISES

web practice

Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTION TO EXERCISES: SOLVING ABSOLUTE VALUE EQUATIONS

1. a. $|x| = 1$, $|x| = 7.2$, and $|x| = \frac{1}{2}$

b. $x = \pm 1$, $x = \pm 7.2$, and $x = \pm\frac{1}{2}$

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2. a. We want the numbers $\underline{x}$, whose $\underline{\text{distance}}$ from $\underline{0}$ is $\underline{5}$. Thus, we want $\underline{x}$ to be $\underline{5}$ or $\underline{-5}$.

b. We want the numbers $\underline{z}$, whose $\underline{\text{distance}}$ from $\underline{0}$ is $\underline{\frac{1}{5}}$. Thus,we want $\underline{z}$ to be $\underline{\frac{1}{5}}$ or $\underline{-\frac{1}{5}}$.

c. We want the numbers $\underline{x}$, whose $\underline{\text{distance}}$ from $\underline{0}$ is $\underline{k}$. Thus, we want $\underline{x}$ to be $\underline{k}$ or $\underline{-k}$.

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3. a. $|x| = 3$ is equivalent to $x = \pm 3$

b. $|t| = 4.2$ is equivalent to $t = \pm 4.2$

4. a. $x = \pm 7$ is equivalent to $|x| = 7$

'$t = \frac{1}{3}$ or $t = -\frac{1}{3}$' is equivalent to $|t| = \frac{1}{3}$

5. For all real numbers x, and for $k \geq 0$, $|x| = k$ is equivalent to $x = \pm k$.

6. The sentence '$|x| = -6$' is not of the form in the theorem, because -6 is a negative number.

7. The sentence '$|x| - 5 = 7$' can be transformed to '$|x| = 12$' by adding 5 to both sides.

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8. a. $|1 + 2x| = 3$ is equivalent to $1 + 2x = \pm 3$
b. $|7x - \frac{1}{2}| = 5$ is equivalent to $7x - \frac{1}{2} = \pm 5$
c. $|x^2 - 8| = 0.4$ is equivalent to $x^2 - 8 = \pm 0.4$

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9. a.

$$\begin{gathered} |1 + 2x| = 3 \\ 1 + 2x = 3 \ \text{ or } \ 1 + 2x = -3 \\ 2x = 2 \ \text{ or } \ 2x = -4 \\ x = 1 \ \text{ or } \ x = -2 \end{gathered}$$

Check: $|1 + 2(1)| = |1 + 2| = |3| = 3$;
$|1 + 2(-2)| = |1 - 4| = |-3| = 3$

b.

$$\begin{gathered} |7x - 5| = 2 \\ 7x - 5 = 2 \ \text{ or } \ 7x - 5 = -2 \\ 7x = 7 \ \text{ or } \ 7x = 3 \\ x = 1 \ \text{ or } \ x = \frac{3}{7} \end{gathered}$$

Check: $|7(1) - 5| = |2| = 2$;
$|7(\frac{3}{7}) - 5| = |3 - 5| = |-2| = 2$

c.

$$\begin{gathered} |4 - 9x| = 1 \\ 4 - 9x = 1 \ \text{ or } \ 4 - 9x = -1 \\ -9x = -3 \ \text{ or } \ -9x = -5 \\ x = \frac{1}{3} \ \text{ or } \ x = \frac{5}{9} \end{gathered}$$

Check: $|4 - 9(\frac{1}{3})| = |4 - 3| = 1$; $|4 - 9(\frac{5}{9})| = |4 - 5| = |-1| = 1$

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10. (a), (c), (d), and (e)are always false; some explanations follow:
Note that $|1 - 4x| + 5 = 0$ is equivalent to $|1 - 4x| = -5$.
Note that $-2|x^2 + 3x - 1| = 8$ is equivalent to $|x^2 + 3x - 1| = -4$.
Note that $|9x + 1| - 5 = -3$ is equivalent to $|9x + 1| = 2$.

11. a.

$$7 - 5|1 - 2x| = -3$$
$$-5|1 - 2x| = -10$$
$$|1 - 2x| = 2$$
$$1 - 2x = 2 \quad \text{or} \quad 1 - 2x = -2$$
$$-2x = 1 \quad \text{or} \quad -2x = -3$$
$$x = -\frac{1}{2} \quad \text{or} \quad x = \frac{3}{2}$$

Check:

$$7 - 5|1 - 2(-\frac{1}{2})| = 7 - 5|1 + 1| = 7 - 5(2) = 7 - 10 = -3$$
$$7 - 5|1 - 2(\frac{3}{2})| = 7 - 5|1 - 3| = 7 - 5|-2| = 7 - 5(2) = 7 - 10 = -3$$

b.

$$-3|2x - 1| - 5 = -4$$
$$-3|2x - 1| = 1$$
$$|2x - 1| = -\frac{1}{3}$$

always false!

c.

$$2|3x - 5| - 1 = 7$$
$$2|3x - 5| = 8$$
$$|3x - 5| = 4$$
$$3x - 5 = 4 \quad \text{or} \quad 3x - 5 = -4$$
$$3x = 9 \quad \text{or} \quad 3x = 1$$
$$x = 3 \quad \text{or} \quad x = \frac{1}{3}$$

Check:

$$2|3(3) - 5| - 1 = 2|9 - 5| - 1 = 2|4| - 1 = 2(4) - 1 = 8 - 1 = 7$$
$$2|3(\frac{1}{3}) - 5| - 1 = 2|1 - 5| - 1 = 2|-4| - 1 = 2(4) - 1 = 8 - 1 = 7$$