25. REVISITING EXPONENTS

exploring expressions like $(-x)^2$, $(-x)^3$, $-x^2$, and $-x^3$

This section explores expressions like $(-x)^2$, $(-x)^3$, $-x^2$, and $-x^3$. The ideas have been discussed before—remember the orders of operation represented by the expressions $(-3)^2$ and -3^2 ? However, this time we'll be using variables instead of constants. By the end of this section, you'll be simplifying complicated-looking problems like $-(-x)^5(-x^3)(-x)^4$ (which equals $-x^{12}$).

rewriting $(-x)^n$ for even powers n

Study the chart below:

expression	expanding ...	summary
$(-x)^2$	$(-x)(-x) = x^2$	$(-x)^2 = x^2$
$(-x)^4$	$(-x)(-x)(-x)(-x) = x^4$	$(-x)^4 = x^4$
$(-x)^6$	$(-x)(-x)(-x)(-x)(-x)(-x) = x^6$	$(-x)^6 = x^6$

In all these cases, there are an even number of factors of -1 , which multiply together to give 1 . Thus, if n is any even number, then $(-x)^n = x^n$. That is:

> Letting EVEN denote any even number $(0, 2, 4, 6, \ldots)$, we have
>
> $$(-x)^{\text{EVEN}} = x^{\text{EVEN}}$$

verbalize: when you raise a number and its opposite to the same even power, you get the same result

Here's an important interpretation of this fact.

Consider a number x and its opposite $-x$.

When you raise x to an even power you get x^{EVEN} .

When you raise $-x$ to the same even power, you get $(-x)^{\text{EVEN}}$.

These two results are the same—they're equal. Thus, we have the following:

> When you raise a number and its opposite to the same even power, then you get the same result.

EXERCISES

1. a. What is 3^2 ? What is $(-3)^2$? Is 3^2 equal to $(-3)^2$?
   b. What is 1^{212} ? What is $(-1)^{212}$. Are they equal?
   c. I'm thinking of a number and its opposite. I'm raising both of them to the fourth power. What (if anything) can you tell me about the results?
   d. I'm thinking of a number. I'm getting a second number by multiplying this first number by -1 . Then, I'm raising both numbers to an even power. What (if anything) can you tell me about the results?

recognizing the pattern $(-x)^{\text{EVEN}}$

Mathematics is largely about recognizing patterns. The expression $(-x)^{\text{EVEN}}$ describes a pattern: the opposite of *something*, raised to an even power. The variable x is being used to represent the *something*. In each example below, look inside the parentheses and think, "the opposite of *what*"?

the expression ...	is of the form ...	where x is ...	and EVEN = ...
$(-y)^{24}$	$(-x)^n$	x is y	EVEN = 24
$(-w)^{110}$	$(-x)^n$	x is w	EVEN = 110
$(-7t)^6$	$(-x)^n$	x is $7t$	EVEN = 6
$(-\frac{1}{2}p)^8$	$(-x)^n$	x is $\frac{1}{2}p$	EVEN = 8
$(-3x)^{14}$	$(-x)^n$	x is $3x$	EVEN = 14
$(\,-(a+2b)\,)^4$	$(-x)^n$	x is $(a+2b)$	EVEN = 4

rewriting $(-x)^{\text{EVEN}}$ *as* x^{EVEN}

Once you've recognized the pattern $(-x)^{\text{EVEN}}$ and identified x and EVEN, then it can be renamed it as x^{EVEN}. Going back to the expressions in the previous table, we have:

$(-y)^{24} = y^{24}$; no parentheses needed on the right-hand side

$(-w)^{110} = w^{110}$; no parentheses needed on the right-hand side

When are parentheses required on the right-hand side?

$(-7t)^6 = (7t)^6$; *the parentheses on the right-hand side are required*! When renaming the pattern $(-x)^{\text{EVEN}}$ as x^{EVEN}, if x is anything other than a single variable (like x or y or w), then you *must* put it inside parentheses when rewriting. The reason in that x—the whole thing!—must be raised to the specified power. Here's a further explanation:

With parentheses in place, we get

$$(7t)^6 = (7t)(7t)(7t)(7t)(7t)(7t)\,,$$

which is what we want. Whatever x is (here, x is $7t$), we need it raised to the 6^{th} power.

However, without parentheses, we would have

$$7t^6 = 7\cdot t\cdot t\cdot t\cdot t\cdot t\cdot t\,,$$

since the power gets done before multiplying by 7. (Remember order of operations!) This is NOT what we want at all. Be careful about this—it's a common algebra mistake.

continuing the examples ...

Continuing the examples:

$(-\frac{1}{2}p)^8 = (\frac{1}{2}p)^8$; remember the parentheses!

$(-3x)^{14} = (3x)^{14}$; remember the parentheses!

$(\,-(a+2b)\,)^4 = (a+2b)^4$; remember the parentheses!

EXERCISES

2. Fill in the chart below:

the expression ...	is of the form ...	where x is ...	and EVEN = ...
$(-k)^{10}$	$(-x)^n$	x is	EVEN =
$(-5x)^{842}$	$(-x)^n$	x is	EVEN =
$(-3.7y)^4$	$(-x)^n$	x is	EVEN =
$(-\frac{2}{9}w)^{16}$	$(-x)^n$	x is	EVEN =
$(\ -(x+y)\ )^8$	$(-x)^n$	x is	EVEN =
$(\ -(3a-5b)\ )^2$	$(-x)^n$	x is	EVEN =

EXERCISES

3. For problems (a)–(d), use a complete mathematical sentence in stating your result. In each case, rewrite the expression using the fact that $(-x)^{\text{EVEN}} = x^{\text{EVEN}}$.

   a. $(-y)^8$

   b. $(-9x)^{206}$

   c. $(-\frac{1}{7}xyz)^6$

   d. $(\ -(\frac{1}{3}x - \frac{2}{5}y)\ )^{104}$

   e. Translate the mathematical equation $(-x)^{\text{EVEN}} = x^{\text{EVEN}}$; that is, use English words to describe what it is saying.

   f. Give the mathematical equation that says: "when you raise a number and its opposite to the same even power, then you get the same result."

comparing the order of operations represented by $(-x)^2$ and $-x^2$

Recall that the expression $(-x)^2$ represents the following sequence of operations:

- take a number;
- find its opposite;
- square the result.

The expression $-x^2$ represents a different sequence of operations:

- take a number;
- square it;
- find the opposite of the result (or, equivalently, multiply by -1).

in general, $(-x)^2$ and $-x^2$ are different numbers

In general, $(-x)^2$ and $-x^2$ are different numbers. They happen to be the same if x is 0, but for all other values of x, $(-x)^2$ is a positive number and $-x^2$ is a negative number. Indeed, $(-x)^2$ and $-x^2$ are opposites. Study the chart below:

x	$(-x)^2$	$-x^2$
0	$(-0)^2 = \mathbf{0}$	$-0^2 = \mathbf{0}$
1	$(-1)^2 = \mathbf{1}$	$-1^2 = -\mathbf{1}$
3	$(-3)^2 = \mathbf{9}$	$-3^2 = -\mathbf{9}$
$\frac{1}{2}$	$(-\frac{1}{2})^2 = \mathbf{\frac{1}{4}}$	$-(\frac{1}{2})^2 = -\mathbf{\frac{1}{4}}$
2.5	$(-2.5)^2 = \mathbf{6.25}$	$-(2.5)^2 = -\mathbf{6.25}$

for all real numbers x, $(-x)^2$ is the opposite of $-x^2$

Recall that 0 is the opposite of 0. So, we can say the following:

> For all real numbers x, $(-x)^2$ and $-x^2$ are opposites.

reading $(-x)^2$ and $-x^2$ aloud:

Mathematics is not designed to be a spoken language, and occasionally we run into things that are inconvenient to read aloud.

unfortunately, they sound the same

Since $(-x)^2$ and $-x^2$ are (in general) different numbers, it would be nice if they sounded different when you read them aloud.

Unfortunately, here's how they're usually read aloud:

reading $(-x)^2$ aloud

$(-x)^2$ is read as:

negative x (slight pause to represent the group), squared

or

the opposite of x (slight pause to represent the group), squared

reading $-x^2$ aloud

$-x^2$ is read as:

negative (slight pause), x squared

or

the opposite of (slight pause), x squared

solving the problem

There's no way that anyone is going to hear the slight pauses in different places! They end up sounding exactly the same. Fortunately, when mathematics is read aloud, it's usually *also* being looked at in written form, so people will see the difference. However, you can also solve the problem like this:

Read $(-x)^2$ as *negative x, the quantity, squared*.

Or, read the parentheses literally:

Read $(-x)^2$ as *open parenthesis, negative x , close parenthesis, squared.*

rewriting $(-x)^n$ for odd powers n

Next, odd powers are explored, and the situation is slightly different. Study the chart below:

expression	expanding ...	summary
$(-x)^3$	$(-x)(-x)(-x) = -x^3$	$(-x)^3 = -x^3$
$(-x)^5$	$(-x)(-x)(-x)(-x)(-x) = -x^5$	$(-x)^5 = -x^5$
$(-x)^7$	$(-x)(-x)(-x)(-x)(-x)(-x)(-x) = -x^7$	$(-x)^7 = -x^7$

In all these cases there are an odd number of factors of -1, which multiply together to give -1. Thus, if n is any odd number, then $(-x)^n = -x^n$. That is,

Letting ODD denote any odd number $(1, 3, 5, 7, \ldots)$, we have:

$$(-x)^{\text{ODD}} = -x^{\text{ODD}}$$

verbalizing:

when you raise a number and its opposite to the same odd power, you get opposites as the result

Again, let's give an important interpretation of this fact.

Consider a number x and its opposite $-x$.

When you raise x to an odd power you get x^{ODD} .

When you raise $-x$ to the same odd power, you get $(-x)^{\text{ODD}}$.

These two results are opposites:

$$\overbrace{(-x)^{\text{ODD}}}^{\text{this number}} \overbrace{=}^{\text{is}} \overbrace{-}^{\text{the opposite of}} \overbrace{x^{\text{ODD}}}^{\text{this number}}$$

When you raise a number and its opposite to the same odd power, you get opposites as the result.

EXERCISES

4. Let $x = 2$.
   a. What do you get when you square x ?
   b. What is the opposite of x ?
   c. What do you get when you square the opposite of x ?
   d. Compare your results to (a) and (c); are they the same, or different?
5. Let $x = 2$.
   a. What do you get when you cube x ?
   b. What is the opposite of x ?
   c. What do you get when you cube the opposite of x ?
   d. Compare your results to (a) and (c); are they the same, or different?
6. Let $x = -2$.
   a. What do you get when you raise x to the fourth power ?
   b. What is the opposite of x ?
   c. What do you get when you raise the opposite of x to the fourth power?
   d. Compare your results to (a) and (c); are they the same, or different?
7. Let $x = -2$.
   a. What do you get when you raise x to the fifth power?
   b. What is the opposite of x ?
   c. What do you get when you raise the opposite of x to the fifth power?
   d. Compare your results to (a) and (c); are they the same, or different?

EXAMPLES

Here are some examples. In each case, you must first recognize the pattern $(-x)^{\text{ODD}}$; the opposite of *something* to an odd power. Then, ask yourself the question, "the opposite of *what*"? Finally, apply the pattern $(-x)^{\text{ODD}} = -x^{\text{ODD}}$ to rename. As before, if x is anything other than a single variable, then you *must* use parentheses on the right-hand side. Remember that the minus sign goes *outside* the parentheses after renaming!

$(-y)^7 = -y^7$; no parentheses needed on the right-hand side

$(-2x)^9 = -(2x)^9$; remember the parentheses!

$(-\frac{2}{7}x^2y)^{13} = -(\frac{2}{7}x^2y)^{13}$; remember the parentheses!

$(\,-(t-2yz)\,)^{103} = -(t-2yz)^{103}$; remember the parentheses!

EXERCISES

8. For problems (a)–(d), use a complete mathematical sentence in stating your result. In each case, rewrite the expression using the fact that $(-x)^{\text{ODD}} = -x^{\text{ODD}}$.
   a. $(-y)^{11}$
   b. $(-9x)^{205}$
   c. $(-\frac{1}{7}xyz)^9$
   d. $(\,-(\frac{1}{3}x - \frac{2}{5}y)\,)^{101}$
   e. Translate the mathematical equation $(-x)^{\text{ODD}} = -x^{\text{ODD}}$; that is, use English words to describe what it is saying.
   f. Give the mathematical equation that says: "when you raise a number and its opposite to the same odd power, then you get opposites as the result."

alternate thought process

Another important point of view in dealing with expressions like $(-x)^2$ and $(-x)^3$ is that the minus sign can be treated as a factor of -1 , and then the exponent law $(ab)^n = a^n b^n$ can be used. Thus, we get:

$$(-x)^2 = (\,(-1)x\,)^2 = (-1)^2 x^2 = x^2$$

and

$$(-x)^3 = (\,(-1)x\,)^3 = (-1)^3 x^3 = -x^3\,.$$

Some people find it easiest to apply this point of view, particularly when the problems get more complicated.

expressions like $(2x)^3$; numbers inside the parentheses

Remember that the exponent law $(ab)^n = a^n b^n$ states that whenever a product is raised to a power, *each factor* must be raised to the power. Thus, for example,

$$(2x)^3 = 2^3 x^3 = 8x^3\,.$$

In particular, $(2x)^3$ is *not* equal to $2x^3$! Be careful about this—it is a common algebra mistake. Notice that $8x^3$ is of the form kx^n where $k = 8$ and $n = 3$.

Also remember the exponent law $(a^m)^n = a^{mn}$: when you have something to a power, to a power, you multiply the exponent. Thus, as a second example, we have:

$$(2x^5y)^3 = 2^3(x^5)^3y^3 = 8x^{15}y^3\,.$$

Notice that the final result ($8x^{15}y^3$) is of the form kx^ny^m , where $k = 8$, $n = 15$, and $m = 3$.

EXERCISES

9. Simplify. Write each expression in the form kx^n . Identify k and n .
   a. $(3x)^2$
   b. $(3x^3)^2$
   c. $(2x^2)^3$
   d. $(2x^3)^3$
   e. $(\frac{1}{2}x)^3$
   f. $(\frac{1}{3}x^2)^3$

EXERCISES

10. Simplify. Write each expression in the form $kx^n y^m$. Identify k , n and m.
    a. $(3xy)^2$
    b. $(3x^3y)^2$
    c. $(2x^2y^3)^3$
    d. $(2xy^2)^3$
    e. $(\frac{1}{2}xy)^2$
    f. $(\frac{1}{2}x^2y^3)^3$

SIGN first; then SIZE

When there is a minus sign inside the parentheses together with a number, then it's usually easiest to figure out the SIGN (plus or minus) of the answer first. If the power is even, then you won't have a minus sign in front of your result. If the power is odd, then you will have a minus sign in front of your result. After taking care of the sign, then take care of the size by raising the number(s) and variable(s) to the appropriate power. Here are some examples:

EXAMPLES

Example: $(-2x)^3$

Notice the minus sign inside the parentheses. The power is odd, so there will be a minus sign in front of the result. Write it down first. You've now accounted for the minus sign inside the parentheses. Then, cube each remaining factor.

Thus, $(-2x)^3 = -2^3x^3 = -8x^3$.

Example: $(-3x^2y)^4$

Notice the minus sign inside the parentheses. The power is even, so there will not be a minus sign in front of the result. You've now accounted for the minus sign inside the parentheses. Then, raise each remaining factor to the fourth power.

Thus, $(-3x^2y)^4 = 3^4(x^2)^4y^4 = 81x^8y^4$.

EXERCISES

11. Simplify. Write without any parentheses. Be sure to write a complete mathematical sentence leading to your final answer.
    a. $(-3x)^2$
    b. $(-3xy)^2$
    c. $(-2x)^3$
    d. $(-2xy)^3$
    e. $(-5x^{-1}y^2)^3$
    f. $(-2x^{-1}y^{-2})^4$

minus signs inside parentheses AND minus signs outside parentheses

As the problems get more complicated, you'll need to pay attention to whether minus signs are *inside* parentheses or *outside* parentheses. Here is the informal thought process:

- If a minus side is *inside* parentheses, look at the exponent, and decide if its even or odd. Rewrite appropriately.
- If a minus sign is *outside* parentheses, then you get a factor of -1 . It may help to write this as (-1) , as illustrated in the following examples.
- At the last step, you'll do the familiar "counting the number of factors of -1": an even number multiplies to 1 , and an odd number to -1 .

EXAMPLES

Here are some examples:

Example: Simplify: $-(-2x)^3$

$-(-2x)^3 = (-1)(-2^3x^3) = 8x^3$

Example: Simplify: $-(-x)^2(-2x)^3$

$-(-x)^2(-2x)^3 = (-1)(x^2)(-2^3x^3) = 8x^5$

Example: Simplify: $-(-x)^5(-x^3)(-x)^4$

$-(-x)^5(-x^3)(-x)^4 = (-1)(-x^5)(-x^3)(x^4) = -x^{12}$

EXERCISES

12. Simplify. Write without any parentheses. Be sure to write a complete mathematical sentence leading to your final answer.
    a. $(-x)^3$
    b. $-(-x)^3$
    c. $(-x)(-x)^2$
    d. $-(-x)^2(-2x)^3$
    e. $(-2x)^3(-x)^4$
    f. $-(-2x)^3(-x)$
    g. $(-5x)^2(-x)^3$
    h. $-(-5x)^2$

EXERCISES
web practice

Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTION TO EXERCISES:
REVISITING EXPONENTS

1. a. $3^2 = 9$ and $(-3)^2 = 9$; they're equal

b. $1^{212} = 1$ and $(-1)^{212} = 1$; they're equal

c. The results are equal.

d. The results are equal.

2.

the expression ...	is of the form ...	where x is ...	and EVEN = ...
$(-k)^{10}$	$(-x)^n$	x is k	EVEN = 10
$(-5x)^{842}$	$(-x)^n$	x is $5x$	EVEN = 842
$(-3.7y)^4$	$(-x)^n$	x is $3.7y$	EVEN = 4
$(-\frac{2}{9}w)^{16}$	$(-x)^n$	x is $\frac{2}{9}w$	EVEN = 16
$(\,-(x+y)\,)^8$	$(-x)^n$	x is $(x+y)$	EVEN = 8
$(\,-(3a-5b)\,)^2$	$(-x)^n$	x is $(3a-5b)$	EVEN = 2

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3. a. $(-y)^8 = y^8$

b. $(-9x)^{206} = (9x)^{206}$

c. $(-\frac{1}{7}xyz)^6 = (\frac{1}{7}xyz)^6$

d. $(\,-(\frac{1}{3}x - \frac{2}{5}y)\,)^{104} = (\frac{1}{3}x - \frac{2}{5}y)^{104}$

e. When you raise a number and its opposite to the same even power, then you get the same result.

f. $(-x)^{\text{EVEN}} = x^{\text{EVEN}}$

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4. a. $2^2 = 4$;

b. the opposite of 2 is -2

c. $(-2)^2 = 4$

d. the results are the same

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5. a. $2^3 = 8$;

b. the opposite of 2 is -2

c. $(-2)^3 = -8$

d. the results are opposites

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6. a. $(-2)^4 = 16$;

b. the opposite of -2 is 2

c. $2^4 = 16$

d. the results are the same

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7. a. $(-2)^5 = -32$;

b. the opposite of -2 is 2

c. $2^5 = 32$

d. the results are opposites

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8. a. $(-y)^{11} = -y^{11}$

b. $(-9x)^{205} = -(9x)^{205}$

c. $(-\frac{1}{7}xyz)^9 = -(\frac{1}{7}xyz)^9$

d. $(\,-(\frac{1}{3}x - \frac{2}{5}y)\,)^{101} = -(\frac{1}{3}x - \frac{2}{5}y)^{101}$

e. When you raise a number and its opposite to the same odd power, then you get opposites as the result.

f. $(-x)^{\text{ODD}} = -x^{\text{ODD}}$

9. Writing in the form kx^n :

a. $(3x)^2 = 3^2x^2 = 9x^2$; $k = 9$, $n = 2$

b. $(3x^3)^2 = 3^2(x^3)^2 = 9x^6$: $k = 9$, $n = 6$

c. $(2x^2)^3 == 2^3(x^2)^3 = 8x^6$: $k = 8$, $n = 6$

d. $(2x^3)^3 = 2^3(x^3)^3 = 8x^9$: $k = 8$, $n = 9$

e. $(\frac{1}{2}x)^3 = (\frac{1}{2})^3x^3 = \frac{1}{8}x^3$: $k = \frac{1}{8}$, $n = 3$

f. $(\frac{1}{3}x^2)^3 = (\frac{1}{3})^3(x^2)^3 = \frac{1}{27}x^6$: $k = \frac{1}{27}$, $n = 6$

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10. Writing in the form kx^ny^m :

a. $(3xy)^2 = 3^2x^2y^2 = 9x^2y^2$; $k = 9$, $n = 2$, $m = 2$

b. $(3x^3y)^2 = 3^2(x^3)^2y^2 = 9x^6y^2$; $k = 9$, $n = 6$, $m = 2$

c. $(2x^2y^3)^3 = 2^3(x^2)^3(y^3)^3 = 8x^6y^9$; $k = 8$, $n = 6$, $m = 9$

d. $(2xy^2)^3 = 2^3x^3(y^2)^3 = 8x^3y^6$; $k = 8$, $n = 3$, $m = 6$

e. $(\frac{1}{2}xy)^2 = (\frac{1}{2})^2x^2y^2 = \frac{1}{4}x^2y^2$; $k = \frac{1}{4}$, $n = 2$, $m = 2$

f. $(\frac{1}{2}x^2y^3)^3 = (\frac{1}{2})^3(x^2)^3(y^3)^3 = \frac{1}{8}x^6y^9$; $k = \frac{1}{8}$, $n = 6$, $m = 9$

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11. a. $(-3x)^2 = 3^2x^2 = 9x^2$

b. $(-3xy)^2 = 3^2x^2y^2 = 9x^2y^2$

c. $(-2x)^3 = -2^3x^3 = -8x^2$

d. $(-2xy)^3 = -2^3x^3y^3 = -8x^3y^3$

e. $(-5x^{-1}y^2)^3 = -5^3(x^{-1})^3(y^2)^3 = -125x^{-3}y^6$

f. $(-2x^{-1}y^{-2})^4 = 2^4(x^{-1})^4(y^{-2})^4 = 16x^{-4}y^{-8}$

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12. a. $(-x)^3 = -x^3$

b. $-(-x)^3 = (-1)(-x^3) = x^3$

c. $(-x)(-x)^2 = (-x)(x^2) = -x^3$

d. $-(-x)^2(-2x)^3 = (-1)(x^2)(-2^3x^3) = (-1)(x^2)(-8x^3) = 8x^5$

e. $(-2x)^3(-x)^4 = (-2^3x^3)(x^4) = (-8x^3)(x^4) = -8x^7$

f. $-(-2x)^3(-x) = (-1)(-2^3x^3)(-x) = (-1)(-8x^3)(-x) = -8x^4$

g. $(-5x)^2(-x)^3 = (5^2x^2)(-x^3) = (25x^2)(-x^3) = -25x^5$

h. $-(-5x)^2 = (-1)(5^2x^2) = (-1)(25x^2) = -25x^2$