24. THE DISTRIBUTIVE LAW

to distribute

One definition of the English verb 'to distribute' is 'to spread out'. The distributive law is one of the most frequently used tools in algebra, and tells how multiplication gets 'spread out' when it interacts when addition—it gives a different order of operations that can be used. The basic statement of the distributive law is deceptively simple. Many important tools, however, are consequences of this law. One of the most famous is 'FOIL', which we'll see in the next section is a memory device for correctly multiplying expressions of the form $(a+b)(c+d)$.

a comfort level with expressions like $(-a)b$

Before presenting the distributive law, it's important to get comfortable with expressions like $(-a)b$ and $-(-a)$, because these make frequent appearances with distributive law work.

a signed variable: $-x$

There are two crucial viewpoints that you should have when you see an expression like $-x$; i.e., a variable, with a minus sign in front of it. For the moment, read $-x$ aloud as 'the opposite of x'.

- Firstly, the symbol $-x$ denotes the opposite of x. If x is positive, then $-x$ is negative. If x is negative, then $-x$ is positive. Study the chart below:

x	$-x$	comment
2	-2	x is positive, so $-x$ is negative
-3	3	x is negative, so $-x$ is positive

- Secondly, the expression $-x$ is equal to $(-1)x$. That is, the minus sign can be thought of as multiplication by -1. Notice how this interpretation is used in the chart below:

x	$-x=(-1)x$	comment
2	$(-1)\cdot 2=-2$	x is positive, so $-x$ is negative
-3	$(-1)\cdot(-3)=3$	x is negative, so $-x$ is positive

reading $-x$ aloud: 'the opposite of x' or 'negative x'

The symbol $-x$ can be read as 'the opposite of x' or 'negative x'. Both are correct, and both are commonplace. Although the phrase 'the opposite of x' is a bit longer, it's also safer for beginning students of algebra. The reason is this: when you say 'negative x' aloud, there is a temptation to think that you're dealing with a negative number (i.e., one that lies to the left of zero on the number line). Not necessarily true! If x is negative, then $-x$ is positive.

If you can say 'negative x' with full knowledge that it's not necessarily a negative number, then go ahead and use this phrase. Otherwise, say 'the opposite of x'.

EXERCISES	1. Let $x = 3$. a. How is the expression '$-x$' read aloud? b. Is x positive or negative? c. Does x lie to the right of zero or to the left of zero? d. Is $-x$ positive or negative? e. Does $-x$ lie to the right of zero or to the left of zero? 2. Let $y = -4$. a. How is the expression '$-y$' read aloud? b. Is y positive or negative? c. Does y lie to the right of zero or to the left of zero? d. Is $-y$ positive or negative? e. Does $-y$ lie to the right of zero or to the left of zero?

Next, we look at products involving signed variables.

products involving signed variables

The product $(-a)b$ can be written in a variety of ways:

$$\overbrace{(-a)b}^{\text{the opposite of } a\text{, times } b} \quad = \quad \overbrace{a(-b)}^{a\text{, times the opposite of } b} \quad = \quad \overbrace{-ab}^{\text{the opposite of } ab}$$

The easiest way to understand this is to think of $-a$ as $(-1)a$. The minus sign can be treated as a factor of -1:

$$\overbrace{(-1)ab}^{\text{(-a)b}} = \overbrace{a(-1)b}^{\text{a(-b)}} = \overbrace{(-1)ab}^{\text{-ab}} .$$

In all these cases, the same three numbers are being multiplied (-1, a, and b) so the result is the same.

use of parentheses and centered dots

People have different preferences as to how much or little they use parentheses, and how much or little they use the centered dot to denote multiplication. There are places where parentheses and centered dots are needed, but in other situations they're optional. You might see any of these:

$$ab = (a)b = a(b) = (a)(b) = a \cdot b = a \cdot (b) = (a) \cdot b = (a) \cdot (b)$$

juxtaposition of variables denotes multiplication

When two variables are juxtaposed (i.e., sitting next to each other), as in the expression ab, the operation between them is multiplication. Thus, ab is the simplest way to denote a multiplied by b.

with signed variables, parentheses are usually required

When a signed variable is involved, parentheses are usually required. To multiply a by $-b$ (in that order) we *can't* just juxtapose them and write $a\,-b$ because this would look like subtraction. Thus, we must write $a(-b)$.

even and odd numbers of factors of -1

When you see opposites in multiplication problems, just treat the minus sign as a factor of -1, and recall these rules:

Any even number of factors of -1 is positive.
For example, $(-1)(-1)(-1)(-1) = 1$.

Any odd number of factors of -1 is negative.
For example, $(-1)(-1)(-1) = -1$.

Conventionally, the minus sign (if there is one) is pulled to the front of the expression, as illustrated in these examples:

$(-a)(-b) = ab$ (two factors of -1)

$-(-a)(b) = ab$ (two factors of -1)

$-a(-b) = ab$ (two factors of -1)

$-(-a)(-b) = -ab$ (three factors of -1)

$(-a)(-b)(-c)(d) = -abcd$ (three factors of -1)

EXERCISES

3. Which of the following are names for $(-c)d$? Circle all correct choices.

$-cd$	$c(-d)$	$(-1)cd$	$c(-1)d$	$d(-c)$
$d-c$	$(-d)c$	$-(cd)$	$-(dc)$	$-(-c)(-d)$

4. Simplify each of the following expressions:
   a. $-(-a)(b)$
   b. $a(-b)(-c)d$
   c. $-(ab)(-c)$
   d. $(-a)(-b)(-c)$
   e. $-a(b)(-c)$
   f. $-a(-b)c$

using exponent laws

You'll also be using exponent laws:

$(-x)(-x)(x) = x^3$

$-(x)(-y)(-xy)(-y) = x^2y^3$

$-x^2(-x^3) = x^5$

$-x(-y)(x) = x^2y$

EXERCISES

5. Simplify.
   a. $-x(-x)(-x)$
   b. $-(-x)(x)$
   c. $x(x^2)(-x^3)$
   d. $-x(x^2)(-x)$
   e. $-x(-y)(xy)(-x^2)$
   f. $-(xy)(-x)(-y^3)$

Now, we're ready to present the distributive law:

THE DISTRIBUTIVE LAW	For all real numbers a, b, and c, $$a(b+c) = ab + ac\,.$$

understanding the statement

The distributive law offers alternate orders of operation that always give the same result.

The expression $a(b+c)$ specifies this order: add b to c; then multiply a by this sum.

The expression $ab+ac$ specifies this order: multiply a and b; multiply a and c; then add these results.

Let's choose some values of a, b and c and verify that the results are the same:

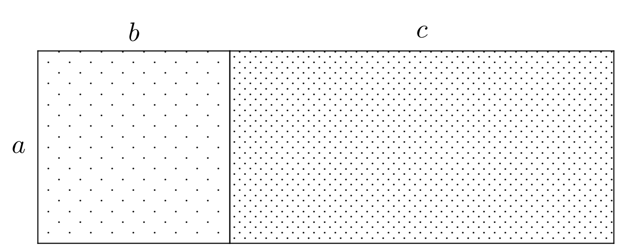
a	b	c	substitution into $a(b+c)$	substitution into $ab+ac$
2	3	4	$2(3+4) = 2(7) = \mathbf{14}$	$(2)(3) + (2)(4) = 6 + 8 = \mathbf{14}$
-2	3	-4	$(-2)(3+(-4)) = (-2)(-1) = \mathbf{2}$	$(-2)(3) + (-2)(-4) = -6 + 8 = \mathbf{2}$
$\frac{1}{2}$	1	$\frac{1}{3}$	$(\frac{1}{2})(1+\frac{1}{3}) = (\frac{1}{2})(\frac{3}{3}+\frac{1}{3}) = (\frac{1}{2})(\frac{4}{3}) = \frac{4}{6} = \mathbf{\frac{2}{3}}$	$(\frac{1}{2})(1) + (\frac{1}{2})(\frac{1}{3}) = \frac{1}{2} + \frac{1}{6} = \frac{3}{6} + \frac{1}{6} = \frac{4}{6} = \mathbf{\frac{2}{3}}$
0.2	-0.4	-0.4	$(0.2)(-0.4+(-0.4)) = (0.2)(-0.8) = -\mathbf{0.16}$	$(0.2)(-0.4) + (0.2)(-0.4) = -0.08 + (-0.08) = -\mathbf{0.16}$

understanding the statement in terms of area

One effective visual way to understand the statement of the distributive law is to use areas.

Form a rectangle with height a, and width $b+c$. The area of this rectangle is height times width: $a(b+c)$.

However, the area can also be found by summing the areas of the two smaller rectangles: $ab+ac$. Thus, $a(b+c) = ab + ac$.



EXERCISES	6. For the given values of a, b, and c, evaluate both $a(b+c)$ and $ab+ac$. Verify that you get the same results. Do the computations without a calculator. a. $a = 3\,,\ b = 5\,,\ c = 7$ b. $a = -2\,,\ b = -1\,,\ c = -3$ c. $a = \frac{1}{3}\,,\ b = -\frac{1}{2}\,,\ c = 1$

using the distributive law

It helps some students first learning to use the distributive law to draw the following arrows:

$$a(b+c) = ab + ac$$

$a(b-c) = ab - ac$

Recall that subtraction is a special kind of addition—so mathematicians don't need to state separate results for subtraction. Addition covers it, like this:

$$\begin{aligned} a(b-c) &= a(b+(-c)) && \text{rewrite subtraction as addition of the opposite} \\ &= ab + a(-c) && \text{use the distributive law} \\ &= ab + (-ac) && \text{write } a(-c) \text{ as } -ac \\ &= ab - ac && \text{rewrite addition of the opposite as subtraction} \end{aligned}$$

Although all the steps are written out here, you need to be able to go from $a(b-c)$ to $ab-ac$ in only one step. The following paragraphs discuss strategies for doing this. First, some terminology:

terms:

In a sum (things added), the things being added are called the *terms*.

a term includes its sign

In the sum $ab + ac$, the terms are ab and ac.

In the expression $ab - ac$, which equals $ab + (-ac)$, the terms are ab and $-ac$. Note that a term includes its sign.

In the expression $a - b + ab - 1$, the terms are a, $-b$, ab, and -1.

EXERCISES	7. List the terms: a. $-x + y - xy + 4$ b. $\frac{1}{2} - x^2y + 3xy - (xy)^2$

decide the sign of each term first

When using the distributive law, it's usually easiest to determine the sign (plus or minus) of each term first. The thought process is detailed below:

$$-a(b-c)\overbrace{=-a(b+(-c))=}^{\text{think about it this way}}-ab+ac$$

The key is to *think* about the expression inside parentheses as $b+(-c)$ (that is, focus on the terms), although you won't write this step down.

- Step 1: Multiply the $-a$ and b; one factor of -1; write down the minus sign, then the ab.
- Step 2: Multiply the $-a$ and $-c$; two factors of -1; write down the plus sign, then the ac.

Go through this thought process on all these problems:

$$\begin{aligned}-a(b+c)&=-ab-ac\\ a(-b+c)&=-ab+ac\\ a(-b-c)&=-ab-ac\\ -a(-b+c)&=ab-ac\\ -a(-b-c)&=ab+ac\\ a(b-c)&=ab-ac\end{aligned}$$

EXERCISES

8. Simplify each of the following in one step:
   a. $-x(y+z)$
   b. $x(-y-z)$
   c. $-x(y-z)$
   d. $-x(-y-z)$

$(a+b)c=ac+bc$

Occasionally you'll see the distributive law with the multiplication on the right instead of the left, like this:

$$(a+b)c=ac+bc$$

All the ideas are the same, for example:

$$\begin{aligned}(a-b)(-c)&=-ac+bc\\ (-a+b)(-c)&=ac-bc\\ (-a-b)c&=-ac-bc\end{aligned}$$

EXERCISES

9. Simplify each of the following in one step:
   a. $(a-b)(-c)$
   b. $(a+b)(-c)$
   c. $(-a+b)c$
   d. $(-a-b)(-c)$
   e. $(a-b)c$

$-(a+b)$

There is a special case that is so simple that it's hard. If you see a minus side outside of a group, with no number there, remember that it's really a factor of -1:

$$-(a+b) \overbrace{= (-1)(a+b) =}^{\text{think, but don't write}} -a-b$$

Every term inside the group gets multiplied by -1, so every term inside the group changes its sign. Thus, for example, we have:

$$-(a-b) = -a+b$$
$$-(-a-b) = a+b$$
$$-(-a+b) = a-b$$

EXERCISES

10. Simplify in one step:
    a. $-(x+y)$
    b. $-(x-y)$
    c. $-(-x+y)$
    d. $-(-x-y)$

problems like $-2x(3y-5z)$*; constants and variables*

When both constants (like 2) and variables (like x) appear in the terms, one little step is added to the thought process. Figure out the sign first, as usual, and write it down. Then, glance through the constants, multiplying as needed, and write this down. Take care of the variable part last. When a constant and a variable appear in the same term, it is conventional to write the constant first. Thus, write $2x$, not $x2$. Here's an example:

$$-2x(3y-5z) = -6xy + 10xz$$

Again, you want to do this in one step. Here's the thought process:

- Step 1: Think about the $-2x$ times $3y$. The sign is $-$; write it down. Glance through the constants (ignoring the signs, because you've already accounted for them) and multiply the 2 and 3; write down the 6. Finally, write down the variable part xy.
- Step 2: Think about the $-2x$ times $-5z$. The sign is $+$; write it down. Glance through the constants (ignoring the signs, because you've already accounted for them) and multiply the 2 and 5; write down the 10. Finally, write down the variable part xz.

EXERCISES

11. Simplify in one step:
    a. $2a(3b-c)$
    b. $-a(-2b+3c)$
    c. $-3a(b-5c)$
    d. $a(-2b+c)$

EXAMPLES This section is concluded with some more complicated problems:

$$-x(3x^2 - y) = -3x^3 + xy$$
$$x^2(-3x - 4xy^5) = -3x^3 - 4x^3y^5$$
$$-xy(4 - 5x^3y^4) = -4xy + 5x^4y^5$$

EXERCISES

12. Simplify:
    a. $-2a(3ab^2 - 5a^3)$
    b. $-a^2(-a + a^2b)$
    c. $3ab(ab - 2b^3a)$
    d. $4a^2b(b - ab)$

EXERCISES *web practice*

Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTION TO EXERCISES: THE DISTRIBUTIVE LAW

1. a. the opposite of x, or negative x
b. positive
c. right of zero
d. negative
e. left of zero

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2. a. the opposite of y, or negative y
b. negative
c. left of zero
d. positive
e. right of zero

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3. $(-c)d = -cd = c(-d) = (-1)cd = c(-1)d = d(-c) = (-d)c = -(cd) - (dc) = -(-c)(-d)$; the only one that isn't a name for $(-c)d$ is $d - c$.

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4. a. $-(-a)(b) = ab$
b. $a(-b)(-c)d = abcd$
c. $-(ab)(-c) = abc$
d. $(-a)(-b)(-c) = -abc$
e. $-a(b)(-c) = abc$
f. $-a(-b)c = abc$

5. a. $-x(-x)(-x) = -x^3$

b. $-(-x)(x) = x^2$

c. $x(x^2)(-x^3) = -x^6$

d. $-x(x^2)(-x) = x^4$

e. $-x(-y)(xy)(-x^2) = -x^4y^2$

f. $-(xy)(-x)(-y^3) = -x^2y^4$

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6. a. $a(b+c) = 3(5+7) = 3(12) = \mathbf{36}$; $ab + ac = (3)(5) + (3)(7) = 15 + 21 = \mathbf{36}$

b. $a(b+c) = -2(-1+(-3)) = -2(-4) = \mathbf{8}$; $ab + ac = (-2)(-1) + (-2)(-3) = 2 + 6 = \mathbf{8}$

c. $a(b+c) = \frac{1}{3}(-\frac{1}{2}+1) = \frac{1}{3}(-\frac{1}{2}+\frac{2}{2}) = \frac{1}{3}(\frac{1}{2}) = \mathbf{\frac{1}{6}}$;
$ab + ac = (\frac{1}{3})(-\frac{1}{2}) + (\frac{1}{3})(1) = -\frac{1}{6} + \frac{1}{3} = -\frac{1}{6} + \frac{2}{6} = \mathbf{\frac{1}{6}}$

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7. a. $-x + y - xy + 4$: the terms are $-x$, y, $-xy$, and 4

b. $\frac{1}{2} - x^2y + 3xy - (xy)^2$: the terms are $\frac{1}{2}$, $-x^2y$, $3xy$, and $-(xy)^2$

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8. $-x(y+z) = -xy - xz$

b. $x(-y-z) = -xy - xz$

c. $-x(y-z) = -xy + xz$

d. $-x(-y-z) = xy + xz$

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9. a. $(a-b)(-c) = -ac + bc$

b. $(a+b)(-c) = -ac - bc$

c. $(-a+b)c = -ac + bc$

d. $(-a-b)(-c) = ac + bc$

e. $(a-b)c = ac - bc$

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10. a. $-(x+y) = -x - y$

b. $-(x-y) = -x + y$

c. $-(-x+y) = x - y$

d. $-(-x-y) = x + y$

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11. a. $2a(3b-c) = 6ab - 2ac$

b. $-a(-2b+3c) = 2ab - 3ac$

c. $-3a(b-5c) = -3ab + 15ac$

d. $a(-2b+c) = -2ab + ac$

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12. a. $-2a(3ab^2 - 5a^3) = -6a^2b^2 + 10a^4$

b. $-a^2(-a + a^2b) = a^3 - a^4b$

c. $3ab(ab - 2b^3a) = 3a^2b^2 - 6a^2b^4$

d. $4a^2b(b - ab) = 4a^2b^2 - 4a^3b^2$