21. EXPONENT LAWS

tools needed for working with exponents

The *exponent laws* are the tools needed for working with expressions involving exponents. They are stated precisely below, and then discussed in the paragraphs that follow.

EXPONENT LAWS

Let x, y, m and n be real numbers, with the following exceptions:

- a base and exponent cannot simultaneously be zero (since 0^0 is undefined);
- division by zero is not allowed;
- for non-integer exponents (like $\frac{1}{2}$ or 0.4), assume that bases are positive.

Then,

$x^m x^n = x^{m+n}$

$x^m x^n = x^{m+n}$

Verbalize: same base, things multiplied, add the exponents

$\frac{x^m}{x^n} = x^{m-n}$

$\frac{x^m}{x^n} = x^{m-n}$

Verbalize: same base, things divided, subtract the exponents

$(x^m)^n = x^{mn}$

$(x^m)^n = x^{mn}$

Verbalize: something to a power, to a power; multiply the exponents

$(xy)^m = x^m y^m$

$(xy)^m = x^m y^m$

Verbalize: product to a power; each factor gets raised to the power

$(\frac{x}{y})^m = \frac{x^m}{y^m}$

$(\frac{x}{y})^m = \frac{x^m}{y^m}$

Verbalize: fraction to a power; both numerator and denominator get raised to the power

the rules are illustrated with positive integers

Although the exponent laws hold for all real numbers (with the stated exceptions), they are illustrated with positive integers.

motivation for $x^m x^n = x^{m+n}$

Notice that

$$x^2 x^3 = \overbrace{x \cdot x}^{\text{two factors}} \cdot \overbrace{x \cdot x \cdot x}^{\text{three factors}} = \overbrace{x \cdot x \cdot x \cdot x \cdot x}^{\text{five factors}} = x^5 = x^{2+3}$$

Here are some examples:

$x^2 x^5 x^3 = x^{2+5+3} = x^{10}$

$x^{\frac{1}{2}} x^{\frac{1}{3}} = x^{\frac{1}{2}+\frac{1}{3}} = x^{\frac{3}{6}+\frac{2}{6}} = x^{\frac{5}{6}}$

$x \cdot y \cdot x^2 \cdot y^2 = (x^1 \cdot x^2)(y^1 \cdot y^2) = x^3 y^3$

$x^{-1} x^5 = x^{-1+5} = x^4$

$(a+b)^2 (a+b)^7 = (a+b)^9$

$2^3 \cdot 2^4 = 2^{3+4} = 2^7 = 128$

EXERCISES	1. Write in the form x^p: a. $x^3x^{-5}x^7$ b. $x^{\frac{1}{5}}x^{\frac{1}{3}}$ c. $x \cdot x^{5.8}$ d. $x^0 \cdot x^1 \cdot x^{-1}$

motivation for $\frac{x^m}{x^n} = x^{m-n}$

Notice that

$$\frac{x^5}{x^2} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x} = x \cdot x \cdot x = x^3 = x^{5-2}$$

Here are some examples:

$\frac{x^7}{x^6} = x^{7-6} = x^1 = x$

$\frac{x^2}{x^9} = x^{2-9} = x^{-7}$

$\frac{x}{x^{-5}} = \frac{x^1}{x^{-5}} = x^{1-(-5)} = x^6$

$\frac{x^2y^3}{xy^5} = \frac{x^2}{x^1} \cdot \frac{y^3}{y^5} = x^{2-1}y^{3-5} = x^1y^{-2} = xy^{-2}$

$\frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}} = x^{\frac{1}{2}-\frac{1}{3}} = x^{\frac{3}{6}-\frac{2}{6}} = x^{\frac{1}{6}}$

$\frac{(a+b)^5}{(a+b)^3} = (a+b)^{5-3} = (a+b)^2$

$\frac{2^{9.4}}{2^{0.5}} = 2^{9.4-0.5} = 2^{8.9}$

flip from top to bottom or bottom to top; change sign of exponent

It's convenient to notice that expressions of the form x^m can be moved from numerator to denominator, or from denominator to numerator, just by changing the sign of the exponent. For example:

$\frac{1}{x^3} = \frac{x^{-3}}{1}$; exponent was positive in denominator; is negative in numerator

$\frac{1}{x^{-3}} = \frac{x^3}{1}$; exponent was negative in denominator; is positive in numerator

$\frac{x^3}{1} = \frac{1}{x^{-3}}$; exponent was positive in numerator; is negative in denominator

$\frac{x^{-3}}{1} = \frac{1}{x^3}$; exponent was negative in numerator; is positive in denominator

Here's a sample proof:

$$\frac{1}{x^{-3}} = 1 \div x^{-3} = 1 \div \frac{1}{x^3} = 1 \cdot \frac{x^3}{1} = \frac{x^3}{1} = x^3 .$$

The other proofs are similar.

EXERCISES	2. Write in the form x^p:
	a. $\dfrac{x^7}{x^2}$
	b. $\dfrac{x}{x^{-4}}$
	c. $\dfrac{x^{\frac{1}{4}}}{x^{\frac{1}{2}}}$
	d. $\dfrac{x^{0.8}}{x^{1.4}}$

motivation for $(x^m)^n = x^{mn}$

Notice that

$$(x^2)^3 = (x^2)(x^2)(x^2) = \overbrace{(x\cdot x)(x\cdot x)(x\cdot x)}^{\text{3 piles, 2 in each}} = x^6 = x^{3\cdot 2}$$

Here are some examples.

$(x^2)^5 = x^{2\cdot 5} = x^{10}$

$(x^3)^{-1} = x^{3\cdot(-1)} = x^{-3}$

$(x^{\frac{1}{2}})^2 = x^{\frac{1}{2}\cdot 2} = x^1 = x$

$\big((a+b)^2\big)^3 = (a+b)^{2\cdot 3} = (a+b)^6$

$(2^{-5})^{-3} = 2^{(-5)(-3)} = 2^{15}$

EXERCISES	3. Write in the form x^p:
	a. $(x^5)^3$
	b. $(x^{-1})^{-4}$
	c. $(x^{\frac{1}{3}})^6$
	d. $(x^2)^{0.6}$

motivation for $(xy)^m = x^m y^m$

Notice that

$$(xy)^3 = (xy)(xy)(xy) = (x\cdot x\cdot x)(y\cdot y\cdot y) = x^3y^3$$

Here are some examples, where the final results are written without any negative exponents:

$(ab)^5 = a^5b^5$

$(xy)^{-4} = x^{-4}y^{-4} = \dfrac{1}{x^4}\cdot\dfrac{1}{y^4} = \dfrac{1}{x^4y^4}$ or $(xy)^{-4} = \dfrac{1}{(xy)^4} = \dfrac{1}{x^4y^4}$

$[(a-1)(b+2)]^3 = (a-1)^3(b+2)^3$

EXERCISES

4. Write in the form $x^p y^p$:
   a. $(xy)^7$
   b. $(xy)^{-6}$
   c. $(xy)^{\frac{1}{2}}$
   d. $(xy)^{1.4}$

motivation for $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$

Notice that

$$\left(\frac{x}{y}\right)^3 = \left(\frac{x}{y}\right)\left(\frac{x}{y}\right)\left(\frac{x}{y}\right) = \frac{x\cdot x\cdot x}{y\cdot y\cdot y} = \frac{x^3}{y^3}$$

Here are some examples:

$\left(\frac{x}{y}\right)^5 = \frac{x^5}{y^5}$

$\left(\frac{a}{b}\right)^{-3} = \frac{a^{-3}}{b^{-3}}$

$\left(\frac{1}{x}\right)^4 = \frac{1^4}{x^4} = \frac{1}{x^4}$

$\left(\frac{a-1}{b+2}\right)^3 = \frac{(a-1)^3}{(b+2)^3}$

EXERCISES

5. Write in the form $\frac{x^m}{y^m}$:
   a. $\left(\frac{x}{y}\right)^7$
   b. $\left(\frac{x}{y}\right)^{-5}$
   c. $\left(\frac{x}{y}\right)^{\frac{1}{3}}$
   d. $\left(\frac{x}{y}\right)^{2.7}$

exponent laws give different orders of operation

It's important to realize that the exponent laws offer a different order of operations that always gives the same result. For example, consider:

$$x^m x^n = x^{m+n}$$

On the left ($x^m x^n$), here is the order of operations:

- raise x to the m power;
- raise x to the n power;
- multiply the two results together.

On the right (x^{m+n}), here is the order of operations:

- add m to n;
- raise x to this power.

Although the order of operations is different, you'll always get the same result. The sentence $x^m x^n = x^{m+n}$ is always true (for all allowable values of x, m, and n).

EXERCISES	6. Describe the order of operations indicated by each side of the equation: a. $\dfrac{x^m}{x^n} = x^{m-n}$ b. $(xy)^m = x^m y^m$

exponent laws can be used left-to-right and right-to-left

It's also important to realize that the exponent laws can be used either left-to-right or right-to-left. For example, consider the law:

$$x^m x^n = x^{m+n}$$

When it is used 'left-to-right,' you would recognize a pattern of the form $x^m x^n$ and rewrite it in the form x^{m+n} , like this:

$$x^2 x^3 = x^{2+3} = x^5$$

When it is used 'right-to-left,' you would recognize a pattern of the form x^{m+n} and rewrite it in the form $x^m x^n$, like this:

$$x^{2+t} = x^2 x^t$$

The exponent laws, as stated, are more frequently used left-to-right. However, be on the lookout for using them in the other direction!

EXERCISES	7. Use the exponent laws 'right-to-left' to rewrite each of the following: a. x^{t+3} b. x^{t-3} c. x^{3t} d. $x^3 y^3$ e. $\dfrac{x^3}{y^3}$

the reason why x^0 must equal 1

With the exponent laws in hand, it can now be shown why x^0 must equal 1. Here's the idea:

$$1 = \overbrace{\frac{5^2}{5^2}}^{\text{numerator and denominator equal}} = \overbrace{5^{2-2}}^{\text{exponent law}} = 5^0$$

Thus, 5^0 must equal 1.

the reason why x^{-1} must equal $\frac{1}{x}$

The exponent laws can also be used to show why x^{-1} must equal $\dfrac{1}{x}$. Here's the idea:

$$\frac{1}{5} = \frac{1}{5^1} = \frac{5^0}{5^1} = \overbrace{5^{0-1}}^{\text{exponent law}} = 5^{-1}$$

Thus, 5^{-1} must equal $\dfrac{1}{5}$.

writing answers using only positive exponents

Sometimes you will be asked to write your final answers with positive exponents only. In this situation, there's a thought process that may cut down your work a bit.

Consider the expression $\frac{x^2}{x^7}$. Although it *could* be simplified like this:

$$\frac{x^2}{x^7} = x^{2-7} = x^{-5} = \frac{1}{x^5}$$

there's a more efficient thought process, as follows.

There are more factors of x in the denominator. How many more? Answer: $7 - 2 = 5$. Thus, after cancellation, there will be 5 factors remaining in the denominator. Notice that you're taking the exponent in the denominator, subtracting the exponent in the numerator, and writing this new exponent in the denominator. Thus, in one step you can go from $\frac{x^2}{x^7}$ to $\frac{1}{x^5}$.

Here are some more examples, where you should be able to go from the original expression to the final expression in one step:

$\frac{x}{x^6} = \frac{1}{x^5}$

$\frac{x^5y^2}{xy^8} = \frac{x^4}{y^6}$

★ Here's the rule that's lurking in the background for this shortcut:

$$\frac{x^m}{x^n} = x^{m-n} = x^{(-1)(n-m)} = (x^{n-m})^{-1} = \frac{1}{x^{n-m}}$$

EXERCISES

8. Simplify in one step. Write without negative exponents.

   a. $\frac{x^3}{x^7}$

   b. $\frac{x^7y^3}{xy^5}$

$\left(\frac{a}{b}\right)^{-2} = \left(\frac{b}{a}\right)^2$

$\left(\frac{a}{b}\right)^{-3} = \left(\frac{b}{a}\right)^3$

$\left(\frac{a}{b}\right)^{-4} = \left(\frac{b}{a}\right)^4$

etc.

One final word about fractions.

Recall from the last section that $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$.

This pattern can now be extended:

$$\begin{aligned}\left(\frac{a}{b}\right)^{-2} &= \frac{1}{(\frac{a}{b})^2} \\ &= 1 \div \left(\frac{a}{b}\right)^2 \\ &= 1 \div \frac{a^2}{b^2} \\ &= 1 \cdot \frac{b^2}{a^2} \\ &= \frac{b^2}{a^2} \\ &= \left(\frac{b}{a}\right)^2\end{aligned}$$

Now that you've seen all the details, you'll never have to go through this argument again. Whenever you see a fraction to a negative power, you can just flip the fraction, and change the sign of the exponent.

EXERCISES

9. Simplify in one step. Write without negative exponents.

   a. $\left(\frac{a}{b}\right)^{-7}$

   b. $\left(\frac{x-1}{y+2}\right)^{-4}$

EXAMPLES

using more than one exponent law

Here are examples that use two or more exponent laws. There is often more than one correct way to approach problems such as these. Final answers are given without negative exponents.

Example: $(x^3)^2(x^5) = x^6x^5 = x^{11}$

Example: $\dfrac{x^{-7}x^5}{x^{-3}} = \dfrac{x^{-2}}{x^{-3}} = x^{-2-(-3)} = x^1 = x$

Example:

$$\begin{aligned}\left(\frac{x^3y^5}{xy^9}\right)^7 &= (x^{3-1}y^{5-9})^7 \\ &= (x^2y^{-4})^7 \\ &= (x^2)^7(y^{-4})^7 \\ &= x^{14}y^{-28} \\ &= \frac{x^{14}}{y^{28}}\end{aligned}$$

EXERCISES

10. Simplify. Write without negative exponents.

    a. $(x^2)^6(x^3)^{-4}$

    b. $\dfrac{x^6x^{-1}}{x^{-3}(x^2)^7}$

    c. $\left(\dfrac{xy}{(x^2y)^{-3}}\right)^5$

EXERCISES
web practice

Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTION TO EXERCISES: EXPONENT LAWS

1. a. $x^3x^{-5}x^7 = x^{3+(-5)+7} = x^5$

b. $x^{\frac{1}{5}}x^{\frac{1}{3}} = x^{\frac{1}{5}+\frac{1}{3}} = x^{\frac{3}{15}+\frac{5}{15}} = x^{\frac{8}{15}}$

c. $x \cdot x^{5.8} = x^1 \cdot x^{5.8} = x^{1+5.8} = x^{6.8}$

d. $x^0 \cdot x^1 \cdot x^{-1} = x^{0+1+(-1)} = x^0 = 1$

2. a. $\dfrac{x^7}{x^2} = x^{7-2} = x^5$

b. $\dfrac{x}{x^{-4}} = \dfrac{x^1}{x^{-4}} = x^{1-(-4)} = x^5$

c. $\dfrac{x^{\frac{1}{4}}}{x^{\frac{1}{2}}} = x^{\frac{1}{4}-\frac{1}{2}} = x^{\frac{1}{4}-\frac{2}{4}} = x^{-\frac{1}{4}}$

d. $\dfrac{x^{0.8}}{x^{1.4}} = x^{0.8-1.4} = x^{-0.6}$

3. a. $(x^5)^3 = x^{15}$

b. $(x^{-1})^{-4} = x^{(-1)(-4)} = x^4$

c. $(x^{\frac{1}{3}})^6 = x^{\frac{1}{3}\cdot 6} = x^2$

d. $(x^2)^{0.6} = x^{2\cdot(0.6)} = x^{1.2}$

4. a. $(xy)^7 = x^7y^7$

b. $(xy)^{-6} = x^{-6}y^{-6}$

c. $(xy)^{\frac{1}{2}} = x^{\frac{1}{2}}y^{\frac{1}{2}}$

d. $(xy)^{1.4} = x^{1.4}y^{1.4}$

5. a. $\left(\dfrac{x}{y}\right)^7 = \dfrac{x^7}{y^7}$

b. $\left(\dfrac{x}{y}\right)^{-5} = \dfrac{x^{-5}}{y^{-5}}$

c. $\left(\dfrac{x}{y}\right)^{\frac{1}{3}} = \dfrac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}}$

d. $\left(\dfrac{x}{y}\right)^{2.7} = \dfrac{x^{2.7}}{y^{2.7}}$

6. a. $\dfrac{x^m}{x^n}$:

- raise x to the m power;
- raise x to the n power;
- divide the first result by the second result.

x^{m-n} :

- take m and subtract n ;
- raise x to this power.

b. $(xy)^m$:

- multiply x and y ;
- raise this product to the m power.

$x^m y^m$:

- raise x to the m power;
- raise y to the m power;
- multiply these two results together.

7. a. $x^{t+3} = x^t x^3$

b. $x^{t-3} = \dfrac{x^t}{x^3}$

c. $x^{3t} = (x^3)^t$

d. $x^3 y^3 = (xy)^3$

e. $\dfrac{x^3}{y^3} = (\dfrac{x}{y})^3$

8. a. $\dfrac{x^3}{x^7} = \dfrac{1}{x^4}$

b. $\dfrac{x^7 y^3}{xy^5} = \dfrac{x^6}{y^2}$

9. a. $(\dfrac{a}{b})^{-7} = (\dfrac{b}{a})^7$

b. $(\dfrac{x-1}{y+2})^{-4} = (\dfrac{y+2}{x-1})^4$

10. a. $(x^2)^6 (x^3)^{-4} = x^{12} x^{-12} = x^{12+(-12)} = x^0 = 1$

b. $\dfrac{x^6 x^{-1}}{x^{-3}(x^2)^7} = \dfrac{x^5}{x^{-3} x^{14}} = \dfrac{x^5}{x^{11}} = \dfrac{1}{x^6}$

c. Here's one correct way:

$$\begin{aligned} \left(\frac{xy}{(x^2y)^{-3}}\right)^5 &= \left(\frac{xy}{(x^2)^{-3}y^{-3}}\right)^5 \\ &= \left(\frac{xy}{x^{-6}y^{-3}}\right)^5 \\ &= (x^{1-(-6)}y^{1-(-3)})^5 \\ &= (x^7y^4)^5 \\ &= (x^7)^5(y^4)^5 \\ &= x^{35}y^{20} \end{aligned}$$