1. THE LANGUAGE OF MATHEMATICS

a hypothetical situation

Imagine the following scenario: you're in math class, and the instructor passes a piece of paper to each student. It is announced that the paper contains *Study Strategies for Students of Mathematics*; you are to read it and make comments. Upon glancing at the paper, however, you observe that it is written in a foreign language that you do not understand!

the importance of language

Is the instructor being fair? Of course not. Indeed, the instructor is probably trying to make a point. Although the *ideas* in the paragraph may be simple, there is no access to the ideas without a knowledge of the *language* in which the ideas are expressed. This situation has a very strong analogy in mathematics. People frequently have trouble understanding mathematical ideas: not necessarily because the ideas are difficult, but because they are being presented in a foreign language—the language of mathematics.

characteristics of the language of mathematics

The language of mathematics makes it easy to express the kinds of thoughts that mathematicians like to express. It is:

- precise (able to make very fine distinctions);
- concise (able to say things briefly);
- powerful (able to express complex thoughts with relative ease).

The language of mathematics *can be learned*, but requires the efforts needed to learn any foreign language. In this book, while you are learning algebra, you will also get extensive practice with mathematical language ideas, to enhance your ability to correctly read, write, speak, and understand mathematics. This first section introduces many concepts that will be studied in more detail throughout the course.

vocabulary versus sentences

Every language has its vocabulary (the words), and its rules for combining these words into complete thoughts (the sentences). Mathematics is no exception. As a first step in discussing the mathematical language, we will make a very broad classification between the 'nouns' of mathematics (used to name mathematical objects of interest) and the 'sentences' of mathematics (which state complete mathematical thoughts).

Why bother making this classification?

The classification of mathematical 'nouns' versus 'sentences' does not typically appear in math books. However, there is tremendous benefit to be derived from this classification of the basic building blocks of mathematics. In the next few paragraphs, comparisons between mathematics and English are explored. The diagram on the opposite page summarizes the language ideas discussed in this section. Come back to this diagram after you've finished reading this section, and it will be much more meaningful to you.

ENGLISH: nouns versus sentences

In English, **nouns** are used to name things we want to talk about (like *people*, *places*, and *things*); whereas **sentences** are used to state complete thoughts. A typical English sentence has at least one noun, and at least one verb. For example, consider the sentence

Carol loves mathematics.

Here, 'Carol' and 'mathematics' are nouns; 'loves' is a verb.

MATHEMATICS:
expressions
versus sentences

The mathematical version of a ‘noun’ will be called an **expression**. Thus, an **expression** is a name given to a mathematical object of interest. Whereas in English we need to talk about people, places, and things, we’ll see that mathematics has much different ‘objects of interest’.

The mathematical version of a ‘sentence’ will also be called a **sentence**. A mathematical sentence, just as an English sentence, must state a complete thought. The table below summarizes the analogy. (Don’t worry for the moment about the *truth* of sentences; this will be addressed later.)

	ENGLISH	MATHEMATICS
name given to an object of interest:	NOUN (person, place, thing) Examples: Carol, Massachusetts, book	EXPRESSION Examples: 5 , $2+3$, $\frac{1}{2}$
a complete thought:	SENTENCE Examples: The capital of Massachusetts is Boston. The capital of Massachusetts is Pittsfield.	SENTENCE Examples: $3+4=7$ $3+4=8$

ideas regarding
expressions:

Let’s discuss the ideas presented in this table, beginning with some ideas regarding expressions.

numbers have
lots of
different names

Since people frequently need to work with *numbers*, these are the most common type of mathematical expression. And, *numbers have lots of different names*. For example, the expressions

$$5 \qquad 2+3 \qquad 10\div 2 \qquad (6-2)+1 \qquad 1+1+1+1+1$$

all *look* different, but are all just different *names* for the same number.

synonyms;
different names for
the same object

This simple idea—that numbers have lots of different names—is extremely important in mathematics. English has the same concept: *synonyms* are words that have the same (or nearly the same) meaning. However, this ‘same object, different name’ idea plays a much more fundamental role in mathematics than in English, as you will see throughout the book.

EXERCISES

Solutions to all exercises are included at the end of each section.

1. The number ‘three’ has lots of different names. Give names satisfying the following properties. There may be more than one correct answer.
    a) the ‘standard’ name
    b) a name using a plus sign, $+$
    c) a name using a minus sign, $-$
    d) a name using a division sign, $\div$
2. Repeat problem 1 with the following numbers: ‘two’, ‘six’, ‘zero’, and ‘one’.

ideas regarding
sentences:
sentences have verbs

Next, some ideas regarding sentences are explored. Just as English sentences have verbs, so do mathematical sentences. In the mathematical sentence ‘ $3+4=7$ ’, the verb is ‘ $=$ ’. If you read the sentence as ‘three plus four is equal to seven’, then it’s easy to ‘hear’ the verb. Indeed, the equal sign ‘ $=$ ’ is one of the most popular mathematical verbs.

truth of sentences

Sentences can be true or false. The notion of **truth** (i.e., the property of being true or false) is of fundamental importance in the mathematical language; this will become apparent as you read the book.

conventions in languages

Languages have *conventions*. In English, for example, it is conventional to capitalize proper names (like ‘Carol’ and ‘Massachusetts’). This convention makes it easy for a reader to distinguish between a common noun (like ‘carol’, a Christmas song) and a proper noun (like ‘Carol’, a person). Mathematics also has its conventions, which help readers distinguish between different types of mathematical expressions. These conventions will be studied throughout the book.

EXERCISES

3. Circle the verbs in the following sentences:
   a) The capital of Massachusetts is Boston.
   b) The capital of Massachusetts is Pittsfield.
   c) $3 + 4 = 7$
   d) $3 + 4 = 8$
4. TRUE or FALSE:
   a) The capital of Massachusetts is Boston.
   b) The capital of Massachusetts is Pittsfield.
   c) $3 + 4 = 7$
   d) $3 + 4 = 8$
5. List several English conventions that are being illustrated in the sentence: ‘The capital of Massachusetts is Boston.’

more examples

Here are more examples, to help explore the difference between sentences and expressions:

EXAMPLE

sentences versus expressions

If possible, classify the entries in the list below as:

- an English noun, or a mathematical expression
- an English sentence, or a mathematical sentence

Try to fill in the blanks yourself before looking at the solutions. In each *sentence* (English or mathematical), circle the verb.

(For the moment, don’t worry about the *truth* of sentences. This issue is addressed in the next example.)

1. cat ________________
2. 2 ________________
3. The word ‘cat’ begins with the letter ‘k’. ________________
4. $1 + 2 = 4$ ________________
5. $5 - 3$ ________________
6. $5 - 3 = 2$ ________________
7. The cat is black. ________________
8. x ________________
9. $x = 1$ ________________
10. $x - 1 = 0$ ________________
11. $t + 3$ ________________
12. $t + 3 = 3 + t$ ________________
13. This sentence is false. ________________
14. $x + 0 = x$ ________________
15. $1 \cdot x = x$ ________________
16. Hat sat bat. ________________

SOLUTIONS:

		HOW TO READ	SOLUTION
1.	cat		English noun
2.	2		mathematical expression
3.	The word ‘cat’ begins with the letter ‘ k ’.		English sentence
4.	$1 + 2 = 4$	‘one plus two equals four’ or ‘one plus two is equal to four’	mathematical sentence
5.	$5 - 3$	‘five minus three’	mathematical expression Note that when you say ‘five minus three’, you have not stated a complete thought.
6.	$5 - 3 = 2$	‘five minus three equals two’ or ‘five minus three is equal to two’	mathematical sentence
7.	The cat is black.		English sentence
8.	x	‘ex’	mathematical expression The letter x (‘ex’) is commonly used in mathematics to represent a number. Such use of letters to represent numbers is discussed in the section **Holding This, Holding That**.
9.	$x = 1$	‘ex equals one’ or ‘ex is equal to one’	mathematical sentence
10.	$x - 1 = 0$	‘ex minus one equals zero’ or ‘ex minus one is equal to zero’	mathematical sentence
11.	$t + 3$	‘tee plus three’	mathematical expression

SOLUTIONS CONTINUED:

		HOW TO READ	SOLUTION
12.	$t + 3 \textcircled{=} 3 + t$	‘tee plus three equals three plus tee’ or ‘tee plus three is equal to three plus tee’	mathematical sentence
13.	This sentence $\textcircled{\text{is}}$ false.		English sentence
14.	$x + 0 \textcircled{=} x$	‘ex plus zero equals ex’ or ‘ex plus zero is equal to ex’	mathematical sentence
15.	$1 \cdot x \textcircled{=} x$	‘one times ex equals ex’ or ‘one times ex is equal to ex’	mathematical sentence The centered dot ‘ $\cdot$ ’ denotes multiplication. Thus, ‘ $1 \cdot x$ ’ is read as ‘one times x ’. You may be used to using the symbol × for multiplication. In algebra, however, the × can get confused with the letter x. (Doesn’t $1 \times x$ look confusing?) Therefore, do NOT use the symbol × for multiplication.
16.	Hat sat bat.		This is not an expression, and not a sentence. Although it has some of the syntax of an English sentence (capital letter at beginning, period at end, a verb), the words have not been used in a proper context to express any meaning. It is nonsensical. It is common for beginning students of mathematics to write ‘nonsensical’ things like this.

sentences state
a complete thought;
expressions don’t

Note that sentences state a complete thought, but nouns and expressions do not. For example, read aloud: ‘ 2 ’. *What about* 2 ? Now read aloud: ‘ $5 - 3 = 2$ ’. This states a complete thought about the number ‘ 2 ’.

Next, the *truth* of sentences is explored:

EXAMPLE

truth of sentences

Consider the entries in the previous example that are *sentences*. Which are true? False? Are there possibilities other than true and false?

Solution:

3.	The word ‘cat’ begins with the letter ‘k’.	FALSE
4.	$1+2=4$	FALSE
6.	$5-3=2$	TRUE
7.	The cat is black.	The truth of this sentence cannot be determined out of context. If the cat being referred to is indeed black, then the sentence is true. Otherwise, it is false.
9.	$x=1$	The letter x represents a number. The truth of this sentence depends upon the number that is chosen for x. If x is replaced by ‘1’, then the sentence becomes the true sentence ‘$1 = 1$’. If x is replaced by ‘2’, then the sentence becomes the false sentence ‘$2 = 1$’. Thus, the sentence ‘$x = 1$’ is SOMETIMES TRUE/SOMETIMES FALSE, depending upon the number that is chosen for x. In sentences such as these, people are often interested in finding the choice(s) that make the sentence true.
10.	$x-1=0$	SOMETIMES TRUE/SOMETIMES FALSE. If x is ‘1’, then the sentence is true. Otherwise, it is false.
12.	$t+3=3+t$	The letter t represents a number. This sentence is TRUE, no matter *what* number is chosen for t. Why? The order that you list the numbers in an addition problem does not affect the result. In other words, commuting the numbers in an addition problem does not affect the result.
13.	This sentence is false.	IF this sentence is true, then it would have to be false. IF this sentence is false, then it would have to be true. So, this sentence is not true, not false, and not sometimes true/sometimes false.
14.	$x+0=x$	This sentence is always TRUE, no matter what number is substituted for x. Adding zero to a number does not change the identity of the number.
15.	$1\cdot x=x$	Recall that the centered dot denotes multiplication. This sentence is always TRUE, no matter what number is substituted for x, since multiplying a number by 1 preserves the identity of the original number.

EXERCISES

6. If possible, classify the entries in the list below as:
   - an English noun, or a mathematical expression
   - an English sentence, or a mathematical sentence

   In each *sentence* (English or mathematical), circle the verb.

   a) Carol ________________
   b) Carol loves mathematics. ________________
   c) The name 'Carol' begins with the letter 'C'. ________________
   d) 7 ________________
   e) $3+4$ ________________
   f) $7=3+4$ ________________
   g) $3+4=7$ ________________
   h) $7=3+5$ ________________
   i) t ________________
   j) $t=2$ ________________
   k) $0=2-t$ ________________
   l) $t-1$ ________________
   m) $t-1=1-t$ ________________
   n) $t+t+t$ ________________
   o) $t-0=t$ ________________
   p) $0=1$ ________________

7. Consider the entries in exercise 6 that are *sentences*. Classify these sentences as: (always) true; (always) false; sometimes true/sometimes false.

definitions in mathematics

With several examples behind us, it is now time to make things more precise.

In order to communicate effectively, people must agree on the meanings of certain words and phrases. When there is ambiguity, confusion can result. Consider the following conversation in a car at a noisy intersection:

Carol: "Turn left!"

Bob: "I didn't hear you. Left?"

Carol: "Right!"

Question: Which way will Bob turn? It depends on how Bob interprets the word 'right'. If he interprets 'right' as the opposite of 'left', then he will turn right. If he interprets 'right' as 'correct,' then he will turn left.

Although there are certainly instances in mathematics where context is used to determine correct meaning, *there is much less ambiguity allowed in mathematics than in English*. The primary way that ambiguity is avoided is by the use of **definitions**. By **defining** words and phrases, it is assured that everyone agrees on their meaning. Here are our first two definitions:

DEFINITION

expression

An *expression* is the mathematical analogue of an English noun; it is a correct arrangement of mathematical symbols used to represent a mathematical object of interest.

Be careful!

Note that an *expression* does NOT state a complete thought. In particular, it does not make sense to ask if an *expression* is true or false.

CAUTION: typical use of the word 'expression' in math books

In most mathematics books, the word 'expression' is never defined, but is used as a convenient catch-all to talk about *anything* (including sentences) to which the author wants to draw attention. In this book, however, **expressions** and **sentences** are totally different entities. They don't overlap. If something is an expression, then it's not a sentence. If something is a sentence, then it's not an expression. Be careful about this.

Next, the definition of a 'mathematical sentence':

DEFINITION
mathematical sentence

A mathematical *sentence* is the analogue of an English sentence; it is a correct arrangement of mathematical symbols that states a complete thought.

Note that it makes sense to ask about the TRUTH of a sentence: Is it true? Is it false? Is it sometimes true/sometimes false?

sentences have verbs

The sentence '$1 + 2 = 3$' is read as '*one plus two equals three*' or '*one plus two is equal to three*'. A complete thought is being stated, which in this case is true. The sentence is 'diagrammed' below:

$$\underbrace{\overbrace{1+2}^{\text{expression}}\ \overbrace{=}^{\text{verb}}\ \overbrace{3}^{\text{expression}}}_{\text{sentence}}$$

how to decide whether something is a sentence

There are two primary ways to decide whether something is a sentence, or not:

- *Read it aloud*, and ask yourself the question: Does it state a complete thought? If the answer is 'yes', then it's a sentence.

  Notice that *expressions do not state a complete thought.* Consider, for example, the number '$1 + 2$'. Say it aloud: '*one plus two*'. Have you stated a complete thought? NO! But, if you say: '$1 + 2 = 4$', then you have stated a complete (false) thought.
- Alternately, you can ask yourself the question: Does it make sense to ask about the TRUTH of this object? Consider again the number '$1 + 2$'. Is '$1 + 2$' true? Is '$1 + 2$' false? These questions don't make sense, because it doesn't make sense to ask about the truth of an expression!

EXERCISES
web practice

8. Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

END-OF-SECTION EXERCISES

For problems 8–15: Classify each entry as a mathematical expression (EXP), or a mathematical sentence (SEN).

Classify the truth value of each entry that is a sentence: (always) true (T); (always) false (F); or sometimes true/sometimes false (ST/SF). The first two are done for you.

(sample)	$1+2$	EXP
(sample)	$1+2=3$	SEN, T
8.	5	________________
9.	$\frac{1}{2}$	________________
10.	$x-1$	________________
11.	$x-1=3$	________________
12.	$1+2+x$	________________
13.	$x \div 3$	________________
14.	$x \div 3 = 2$	________________
15.	$1+2+x=x+1+2$	________________

16. Use the English noun 'Julia' in three sentences: one that is true, one that is false, and one whose truth cannot be determined without additional information.
17. Use the mathematical expression '3' in three sentences: one that is true, one that is false, and one whose truth cannot be determined without additional information.
18. Use the mathematical expression 'x' in three sentences: one that is always true, one that is always false, and one whose truth cannot be determined without additional information.

SECTION SUMMARY
THE LANGUAGE OF MATHEMATICS

NEW IN THIS SECTION	HOW TO READ	MEANING
expression		The mathematical analogue of an English noun; a correct arrangement of mathematical symbols used to represent a mathematical object of interest. An expression does NOT state a complete thought; it does not make sense to ask if an expression is true or false. Most common expression types: numbers, sets, functions.
sentence		The mathematical analogue of an English sentence; a correct arrangement of mathematical symbols that states a complete thought. It makes sense to ask if a sentence is true, false, sometimes true/sometimes false.
$x \cdot y$	x times y	a centered dot between numbers (or letters representing numbers) denotes multiplication