12. MULTIPLYING FRACTIONS

multiplying fractions

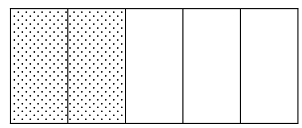
This section discusses multiplication and division of fractions, and related concepts.

Multiplying fractions is easy: just multiply the numerators, and multiply the denominators. That is,

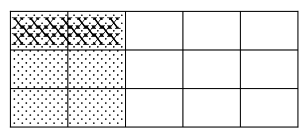
$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}\,.$$

To see why multiplication of fractions is defined in this way, consider the problem $\frac{1}{3} \cdot \frac{2}{5}$. That is, we want $\frac{1}{3}$ of $\frac{2}{5}$:

First, shade $\frac{2}{5}$:



Now, we want $\frac{1}{3}$ of this shaded part:



Notice that the entire object has been divided into 15 equal parts, and we want 2 of these, so $\frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}$.

EXERCISES

1. Make a sketch that illustrates the multiplication of fractions problem $\frac{1}{4} \cdot \frac{2}{3}$.
2. Multiply.
   a. $\frac{1}{7} \cdot \frac{2}{3}$
   b. $\frac{2}{11} \cdot \frac{3}{5}$
   c. $\frac{a}{b} \cdot \frac{c}{d}$
   d. $\frac{1}{x} \cdot \frac{y}{1}$
   e. $\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{5}$
   f. $\frac{0}{5} \cdot \frac{7}{4}$

reciprocal

The *reciprocal* of a number is found by taking 1 over the number. That is,

$$\text{the reciprocal of } x \text{ is } \frac{1}{x}$$

Alternately, the reciprocal can be defined as the unique number which multiplies the original number to give 1:

$$(\text{a number}) \times (\text{its reciprocal}) = 1$$

EXAMPLES

finding reciprocals

As the following examples illustrate, finding a reciprocal is as simple as a 'flip'!

The reciprocal of 3 is $\frac{1}{3}$.

The reciprocal of $\frac{1}{3}$ is 3. Being a reciprocal is a reciprocal relationship: if a is the reciprocal of b, then b is the reciprocal of a.

The reciprocal of 2.1 is $\frac{1}{2.1}$.

The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$ (since $\frac{2}{3} \cdot \frac{3}{2} = 1$).

In general, the reciprocal of $\dfrac{a}{b}$ is $\dfrac{b}{a}$, as long as a and b are both nonzero.

The reciprocal of 1 is $\frac{1}{1} = 1$, so the number 1 is its own reciprocal.

The reciprocal of -1 is $\frac{1}{-1} = -1$, so -1 is also its own reciprocal.

The number 0 does not have a reciprocal, since $\frac{1}{0}$ is not defined.

EXERCISES

3. Find the reciprocals of the following numbers, if they exist.
   a. 5
   b. $\dfrac{1}{5}$
   c. 3.7
   d. $\dfrac{a}{b}$
   e. $\dfrac{a+1}{b+2}$
   f. $\dfrac{0}{3}$

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dividing fractions: multiply by the reciprocal

Every division problem is a multiplication problem in disguise: to divide by a number means to multiply by its reciprocal.

That is, x divided by y is the same as x times the reciprocal of y.

In symbols,

$$\overbrace{\frac{x}{y}}^{x \text{ divided by } y} \quad \overbrace{=}^{\text{is}} \quad \overbrace{x \cdot \frac{1}{y}}^{x \text{ times the reciprocal of } y}$$

Here's what it looks like with fractions:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

For example,

$$\frac{1}{3} \div \frac{2}{5} = \frac{1}{3} \cdot \frac{5}{2} = \frac{5}{6}\,.$$

As a second (horrendous-looking!) example,

$$\frac{\frac{2}{7}}{\frac{3}{5}} = \frac{2}{7} \div \frac{3}{5} = \frac{2}{7} \cdot \frac{5}{3} = \frac{10}{21}\,.$$

EXERCISES

4. Do the following division problems:
   a. $\frac{1}{2} \div \frac{3}{5}$
   b. $3 \div \frac{1}{3}$
   c. $\frac{1}{3} \div 3$
   d. $\frac{c}{d} \div \frac{a}{b}$
   e. $\frac{2}{\frac{3}{5}}$
   f. $\frac{\frac{2}{7}}{3}$

every fraction has three signs associated with it

Every fraction has three signs associated with it: the sign of the numerator (plus or minus), the sign of the denominator (plus or minus), and the sign of the fraction itself (plus or minus). The sign of the fraction goes in front of the fraction, like this:

$$\overbrace{-}^{\text{here}}\frac{N}{D}$$

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EXAMPLES

signs of fractions

Here are some examples:

$\frac{3}{5}$: the numerator is positive, the denominator is positive, the fraction sign is positive

$-\frac{3}{5}$: the numerator is positive, the denominator is positive, the fraction sign is negative

$\frac{-3}{5}$: the numerator is negative, the denominator is positive, the fraction sign is positive

$\frac{3}{-5}$: the numerator is positive, the denominator is negative, the fraction sign is positive

$\frac{-3}{-5}$: the numerator is negative, the denominator is negative, the fraction sign is positive

$-\frac{-3}{5}$: the numerator is negative, the denominator is positive, the fraction sign is negative

$-\frac{-3}{-5}$: the numerator is negative, the denominator is negative, the fraction sign is negative

$-\frac{3}{-5}$: the numerator is positive, the denominator is negative, the fraction sign is negative

EXERCISES

5. Determine all three signs for each fraction:
   a. $\frac{2}{7}$
   b. $-\frac{-2}{7}$
   c. $-\frac{2}{-7}$
   d. $\frac{-2}{-7}$
   e. $-\frac{-2}{-7}$

replacing messy fractions with simpler ones

Some fractions involving lots of signed numbers can get pretty messy looking. For example, this fraction looks pretty complicated:

$$-\frac{(-1)(-2)(3)(-1)}{(-5)(-1)(-7)}$$

The good news is that you can always replace any fraction with a fraction that has exactly one minus sign, or no minus sign. Much simpler! Here are the ideas that you need:

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multiplication of signed numbers

Recall the following properties of multiplication of signed numbers:

(positive)(positive) = positive; for example, $(2)(3) = 6$

(positive)(negative) = negative; for example, $(2)(-3) = -6$ and $(-2)(3) = -6$

(negative)(negative) = positive; for example, $(-2)(-3) = 6$

The same pattern holds for division:

$\dfrac{\text{positive}}{\text{positive}}$ = positive; for example, $\frac{6}{2} = 3$

$\dfrac{\text{positive}}{\text{negative}}$ = negative; for example, $\frac{6}{-2} = -3$

$\dfrac{\text{negative}}{\text{positive}}$ = negative; for example, $\frac{-6}{2} = -3$

$\dfrac{\text{negative}}{\text{negative}}$ = positive; for example, $\frac{-6}{-2} = 3$

EXERCISES

6. Suppose that x is positive and y is negative. Decide whether each of the following numbers is positive or negative. Remember that $x^2 = x \cdot x$ and xy means x times y.

   a. x^2

   b. xy

   c. y^2

   d. $\dfrac{x}{x}$

   e. $\dfrac{x}{y}$

   f. $\dfrac{y}{y}$

   g. $\dfrac{2}{x}$

   h. $\dfrac{y}{2}$

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count the number of minus signs:

even number, +;

odd number, −

But suppose you have more than two numbers? Just use the rules more than once. For example,

$$\overbrace{(+)(-)}(-) = \overbrace{(-)}(-) \qquad \text{the first } (+)(-) \text{ gives a } (-)$$
$$= (+)$$

Certainly you don't want to go through this thought process all the time, and you don't have to. When there are more than two numbers being multiplied, it just boils down to counting the number of minus signs: any even number of minus signs collapses to a plus sign, and any odd number of minus signs collapses to a minus sign. You figure out the sign (plus or minus) of the result, and then mentally 'throw away' all the minus signs and compute as usual.

Here's an example. Returning to the earlier messy-looking fraction:

$$-\frac{(-1)(-2)(3)(-1)}{(-5)(-1)(-7)}$$

It has 7 minus signs; 7 is odd; so they all collapse to a single minus sign. Thus,

$$-\frac{(-1)(-2)(3)(-1)}{(-5)(-1)(-7)} = -\frac{6}{35}\,.$$

the single minus sign can go in three different places

Make sure you realize that the single minus sign can go in three different places: in the numerator, in the denominator, or in front of the fraction. That is,

$$-\frac{N}{D} = \frac{-N}{D} = \frac{N}{-D}\,.$$

It's most common to put the minus sign in front or on top.

EXERCISES

7. Simplify:
   a. $-\dfrac{(-1)(2)(-3)}{(-5)(7)(-1)(-1)}$
   b. $\dfrac{(-2)(-3)(-1)}{(-5)(-7)}$
   c. $-\dfrac{-3}{5(-1)}$
8. Why do you suppose the author named this section 'Multiplying Fractions' instead of 'Multiplying and Dividing Fractions'?

rewriting the pattern $a \cdot \frac{b}{c}$

One pattern that arises frequently in working with fractions is $a \cdot \frac{b}{c}$. It's important to realize that this expression can be written in many different ways:

$$
\begin{aligned}
a \cdot \frac{b}{c} &= \frac{ab}{c} = \frac{ba}{c} = b \cdot \frac{a}{c} = ab \cdot \frac{1}{c} \\
&= ba \cdot \frac{1}{c} = a \cdot \frac{1}{c} \cdot b = \frac{1}{c} \cdot ba \\
&= b \cdot \frac{1}{a} \cdot c = \frac{a}{c} \cdot b = \frac{1}{c} \cdot ab = \cdots
\end{aligned}
$$

Note that a factor in the numerator can optionally be centered next to the fraction. Or, a factor centered next to a fraction can be moved into the numerator. However, a factor in the denominator must stay in the denominator.

EXERCISES

9a. Give three different names for the expression $b \cdot \frac{c}{a}$.

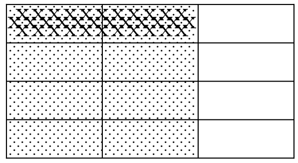
9b. Give three different names for the expression $\frac{ab}{cd}$.

EXERCISES
web practice

Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTIONS TO EXERCISES: MULTIPLYING FRACTIONS

1.



$$\frac{1}{4} \cdot \frac{2}{3} = \frac{2}{12}$$

2. a. $\dfrac{1}{7} \cdot \dfrac{2}{3} = \dfrac{2}{21}$

b. $\dfrac{2}{11} \cdot \dfrac{3}{5} = \dfrac{6}{55}$

c. $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$

d. $\dfrac{1}{x} \cdot \dfrac{y}{1} = \dfrac{y}{x}$

e. $\dfrac{1}{2} \cdot \dfrac{1}{3} \cdot \dfrac{1}{5} = \dfrac{1}{30}$

f. $\dfrac{0}{5} \cdot \dfrac{7}{4} = 0 \cdot \dfrac{7}{4} = 0$

3. a. The reciprocal of 5 is $\dfrac{1}{5}$.

b. The reciprocal of $\dfrac{1}{5}$ is 5.

c. The reciprocal of 3.7 is $\dfrac{1}{3.7}$.

d. The reciprocal of $\dfrac{a}{b}$ is $\dfrac{b}{a}$.

e. The reciprocal of $\dfrac{a+1}{b+2}$ is $\dfrac{b+2}{a+1}$.

f. The number $\dfrac{0}{3} = 0$ has no reciprocal.

4. a. $\dfrac{1}{2} \div \dfrac{3}{5} = \dfrac{1}{2} \cdot \dfrac{5}{3} = \dfrac{5}{6}$

b. $3 \div \dfrac{1}{3} = 3 \cdot 3 = 9$

c. $\dfrac{1}{3} \div 3 = \dfrac{1}{3} \cdot \dfrac{1}{3} = \dfrac{1}{9}$

d. $\dfrac{c}{d} \div \dfrac{a}{b} = \dfrac{c}{d} \cdot \dfrac{b}{a} = \dfrac{cb}{da}$

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e. $\dfrac{2}{\frac{3}{5}} = 2 \div \dfrac{3}{5} = 2 \cdot \dfrac{5}{3} = \dfrac{2}{1} \cdot \dfrac{5}{3} = \dfrac{10}{3}$

f. $\dfrac{\frac{2}{7}}{3} = \dfrac{2}{7} \div 3 = \dfrac{2}{7} \cdot \dfrac{1}{3} = \dfrac{2}{21}$

5. a. $\dfrac{2}{7}$: numerator is positive; denominator is position; fraction sign is positive

b. $-\dfrac{-2}{7}$: numerator is negative; denominator is positive; fraction sign is negative

c. $-\dfrac{2}{-7}$: numerator is positive; denominator is negative; fraction sign is negative

d. $\dfrac{-2}{-7}$: numerator is negative; denominator is negative; fraction sign is positive

e. $-\dfrac{-2}{-7}$: numerator is negative; denominator is negative; fraction sign is negative

6. a. x^2 is positive

b. xy is negative

c. y^2 is positive

d. $\dfrac{x}{x}$ is positive

e. $\dfrac{x}{y}$ is negative

f. $\dfrac{y}{y}$ is positive

g. $\dfrac{2}{x}$ is positive

h. $\dfrac{y}{2}$ is negative

7. a. $-\dfrac{(-1)(2)(-3)}{(-5)(7)(-1)(-1)} = \dfrac{6}{35}$

b. $\dfrac{(-2)(-3)(-1)}{(-5)(-7)} = -\dfrac{6}{35}$

c. $-\dfrac{-3}{5(-1)} = -\dfrac{3}{5}$

8. To a mathematician, there is only multiplication, since every division problem is a multiplication problem in disguise: $x \div y = x \cdot \frac{1}{y}$.

9. Here are some possible answers:

a. $b \cdot \dfrac{c}{a} = \dfrac{c}{a} \cdot b = \dfrac{bc}{a} = \dfrac{cb}{a} = c \cdot \dfrac{b}{a} = \dfrac{b}{a} \cdot c = cb \cdot \dfrac{1}{a} = bc \cdot \dfrac{1}{a} = \cdots$

b. $\dfrac{ab}{cd} = a \cdot \dfrac{b}{cd} = b \cdot \dfrac{a}{cd} = ab \cdot \dfrac{1}{cd} = \dfrac{b}{cd} \cdot a = \dfrac{a}{cd} \cdot b = \dfrac{1}{cd} \cdot ab = \dfrac{1}{c} \cdot \dfrac{a}{d} \cdot b = \dfrac{1}{d} \cdot \dfrac{b}{c} \cdot a = \cdots$

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