COURSE OBJECTIVES LIST: PRECALCULUS

Precalculus Honors is offered.

PREREQUISITES: All skills from Algebra I, Geometry, and Algebra II are assumed. A prerequisites test is given during the first week of class to assess knowledge of these prerequisite skills and to locate deficiencies.

COURSE BOOK DESCRIPTION:

In Precalculus, the set of skills needed for success in Calculus is completed. Students become fluent in the language of functions and many function applications are explored. A large component of the course explores trigonometry, interweaving the unit circle and right triangle viewpoints. A study of vectors, polar coordinates, parametric equations and partial fractions completes the preparation for more advanced mathematics. Completion of Algebra II and permission from the mathematics department are prerequisites for enrollment. An honors section encourages more creative, critical and in-depth study of these topics.

The course objectives are elaborated as follows. The order in which the objectives are listed is not necessarily the order in which they will be taught.

FUNCTIONS:

FCT1. Evaluate complex expressions involving function notation, including difference quotients: $\frac{f(x+h)-f(x)}{h}$.

FCT2. Introduce the idea of tangent line as the best linear approximation to a curve at a point (when it exists). Investigate the slope of a (non-vertical) tangent line: talk about what happens to the difference quotient $\frac{f(x+h)-f(x)}{h}$ as h approaches zero.

FCT3. Determine the domain of a function from its formula. Use the notation $\text{dom}(f)$ and $\text{ran}(f)$ to denote the domain and range of f , respectively. (This will draw on tools developed in Algebra II: basic knowledge of function behavior, and sentence-solving skills.)

FCT4. Define: even and odd functions. Explain the definitions: e.g., a function f is even if and only if for all x , $f(x) = f(-x)$. Explain in words: when inputs are opposites, outputs are the same. Explain the graphical consequences.

FCT5. READ A WIDE VARIETY OF INFORMATION from the graph of $y = f(x)$: domain and range; x and y intercept(s); function values; sets of x-values satisfying certain properties, like $\{x \mid f(x) > 2\}$ and $\{x \mid |f(x) - \ell| < \epsilon\}$. Report answers using correct notation, being careful to distinguish between numbers and sets.

FCT6. Determine, from a graph, interval(s) on which f increases, decreases, is constant. Define local max/min versus global max/min. Be able to estimate all this information from a calculator graph.

FCT7. Given a formula or graph, apply multi-step transformations involving any combination of horizontal/vertical translation; reflections about the x-axis and y-axis; vertical scaling; horizontal compression and elongation; absolute value transformation. Give the resulting formula or graph. In particular, be able to list transformations that take you from $y = f(x)$ to $y = a \cdot f(bx + c) + d$.

FCT8. RECOGNIZE the following notation for horizontal compressions and elongations (let $k > 1$):

'horizontal compression by a factor of k' takes a point (a, b) to the point $(\frac{a}{k}, b)$. For example, applying 'horizontal compression by a factor of 2' to $y = f(x)$ yields the equation $y = f(2x)$ and takes (a, b) to $(\frac{a}{2}, b)$.

'horizontal elongation by a factor of k' takes a point (a, b) to the point (ka, b). For example, applying 'horizontal elongation by a factor of 3' to $y = f(x)$ yields the equation $y = f(\frac{x}{3})$ and takes (a, b) to $(3a, b)$.

FCT9. Define one-to-one function; determine if a function is one-to-one (from a formula; from a graph).

FCT10. Given a function, find its inverse (if one exists). Use the notation f^{-1} for the inverse of f. Emphasize that f^{-1} does NOT denote $\frac{1}{f}$.

FCT11. Explain the relationship between a function and its inverse: $f\big(f^{-1}(x)\big) = x$ for all $x \in \text{ran}(f)$ and $f^{-1}\big(f(x)\big) = x$ for all $x \in \text{dom}(f)$. Explain the relationship in terms of points: point (a, b) is on the graph of f if and only if (b, a) is on the graph of f^{-1}.

FCT12. Explain that the points (a, b) and (b, a) are mirror images about the line $y = x$ (when the scales on the x-axis and y-axis are the same). Graph the inverse from a graph of a one-to-one function. Explain the relationship between the domains and ranges of f and f^{-1}.

FCT13. Given the graph of a one-to-one function f, read off information about f^{-1}.

FCT14. Use the 'Test Point Method' for solving inequalities in one variable. Explain the key idea behind this method: there are only two types of places where a function can change its sign—at a break in the graph, or where it equals zero. Locate all such place(s); test the resulting subinterval(s).

Solve a wide variety of inequalities using the test point method. Report solution sets using correct set notation. Emphasize the graphical interpretations of solution sets.

FCT15. Graph a wide variety of equations and inequalities in two variables. Distinguish between boundaries that are included (solid line) and not included (dashed line). (For example, graph $y > x^2$ and $x - 3y < 1$.)

FCT16. Determine, from both a graph and an equation, if there is symmetry about the x-axis, y-axis, origin. Explain the definitions: e.g., a graph is symmetric about the x-axis if and only if whenever (a, b) is on the graph, so is $(a, -b)$.

POLYNOMIALS: Build on the skills begun in Algebra II:

- FCT17. For a polynomial with real number coefficients, any non-real zeros must occur in complex conjugate pairs.
- FCT18. Explain what is meant by the multiplicity of a zero, including the graphical interpretation.
- FCT19. Explain the Fundamental Theorem of Algebra: every polynomial of degree n has exactly n zeros in $\mathbb{C}$ (counting multiplicity). Look at a special case: every polynomial of degree n with real coefficients can be factored into linear and irreducible quadratic factors.
- FCT20. Factor a polynomial with real number coefficients into linear and irreducible quadratic factors.
- FCT21. USE the Confinement Theorem (below) as a tool to help determine an appropriate calculator viewing window and locate zeros. This theorem will be provided when needed.

  The CONFINEMENT THEOREM: The polynomial equation

  $$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0 \ , \quad a_n \neq 0$$

  has at most n real solutions (by the Fundamental Theorem of Algebra), and they are contained in the interval $[-K, K]$, where $K = \frac{nM}{|a_n|}$ and M is the largest of $|a_n|, |a_{n-1}|, \ldots, |a_0|$.
- FCT22. DETERMINE the equation of a polynomial satisfying specified conditions, including multiplicity considerations.

RATIONAL FUNCTIONS: Build on the skills begun in Algebra II:

- FCT23. Give the formula of a function having specified asymptotes; give a graph that exhibits specified asymptote behavior.

EXPONENTIAL and LOGARITHMIC FUNCTIONS: Build on the skills begun in Algebra II:

- FCT24. Write exponential functions in the forms b^t and e^{kt}. Understand the effects of changing the base. Emphasize that exponential functions are one-to-one, hence have inverses.
- FCT25. Emphasize that exponential functions have a constant factor of change: for $f(t) = b^{kt}$, $f(t + \Delta t) = b^{k\Delta t} f(t)$. Contrast this with the behavior of linear functions.
- FCT26. Emphasize that logarithmic functions are one-to-one, hence have inverses. Logarithmic functions should be understood from two viewpoints: as the inverse of exponential functions; and as *exponents* ($\log_b x$ is the exponent that b must be raised to, to get x).

FCT27. From the graph of $y = b^x$, students should be able to approximate values of $\log_b x$; from the graph of $y = \log_b x$, students should be able to approximate values of b^x.

FCT28. SOLVE exponential growth and decay problems (e.g., half-life, doubling time).

PIECEWISE-DEFINED FUNCTIONS:

FCT29. Solve equations and inequalities involving piecewise-defined functions. The pieces may draw on the following function types: polynomial, rational, exponential, logarithmic, trigonometric, radical.

TRIGONOMETRY:

TRIG1. Understand both degree and radian measure, and be able to convert between angle measures. Be aware of angle mode when using a calculator. Know what is meant by "standard position" of an angle; be able to find the "reference angle" for a given angle.

TRIG2. Find the length of an arc, and the area of a sector.

TRIG3. Give unit circle definitions of sine and cosine; define all other trig functions in terms of sine and cosine. Also give right-triangle definitions for all the trig functions.

Students should understand how the unit circle and right triangle definitions relate to each other: when they see a right triangle, they should be able to scale it (divide by the length of the hypotenuse) so it fits in the unit circle; when they look at a point on the unit circle, a right triangle should jump out at them.

TRIG4. Determine the signs (plus or minus) of all the trig functions in all quadrants. State and use properties of trig functions that are direct consequences of the unit circle definition: e.g., $-1 \le \sin x \le 1$, $\sin(-x) = -\sin x$.

TRIG5. Determine exact values of trig functions of special angles (multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$). In particular, students should memorize the 45°-45°-90° and 30°-60°-90° triangles.

TRIG6. Solve right triangles.

TRIG7. State and use the Law of Sines. Students should be aware of conditions under which no triangle exists; a unique triangle exists; two triangles exist.

TRIG8. State and use the Law of Cosines.

TRIG9. Find the area of (arbitrary) triangles.

TRIG10. Define periodic function. State the periods of all the trig functions.

TRIG11. Graph all the trigonometric functions, and read off important information: e.g., domain, range, asymptote(s), period, intercept(s). Also graph "transformed" trig functions, using all the shifting/reflecting/scaling/etc. techniques from earlier.

TRIG12. Write an equation from a graph (the answer will not be unique).

Read amplitude/period/frequency/phase shift information from $y = A\sin(Bx+C)$ (or cosine), and graph.

TRIG13. Explain arcsine (arccosine, arctangent) as the inverse of an appropriately-restricted sine (cosine, tangent). Give precise definitions of arcsine, arccosine, and arctangent (see next item for an example).

Recognize common notations for the inverse trig functions: $\sin^{-1} x$ and $\arcsin x$, etc.

TRIG14. State and use the precise relationship between sine and arcsine (cosine/arccosine; tangent/arctangent): for example,

$$y = \arcsin x \quad \text{if and only if} \quad \left(\sin y = x \ \text{ and } \ -\frac{\pi}{2} \le y \le \frac{\pi}{2}\right) .$$

In particular, be able to verbalize: "$\arcsin x$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x."

TRIG15. Graph the inverse trig functions.

TRIG16. Explain how many identities follow naturally from the periodic nature of the trigonometric functions (e.g., $\sin(x+2\pi) = \sin x$). State the Pythagorean Identities.

TRIG17. Verify trigonometric identities: write a list of equivalent equations, ending with one that is always true, OR, verify that one side of the equation is always equal to the other side.

TRIG18. State and use the sum and difference formulas for sine and cosine.

TRIG19. State and use the double angle formulas for sine and cosine.

TRIG20. Solve simple trigonometric equations: e.g., find all $t \in [0, 2\pi]$ for which $\cos\frac{t}{2} = -\frac{1}{2}$.

CONIC SECTIONS:

CONIC1. Identify the graph of an equation of the form $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ as a circle, ellipse, hyperbola, parabola. Graph in the case where $B=0$.

CONIC2. PARABOLA: Give geometric definition; derive equation of parabola in standard form; graph parabolas (unrotated); write equations of parabolas.

CONIC3. ELLIPSE and HYPERBOLA: Give geometric definitions; derive equations in standard form; graph (unrotated); write equations of ellipses.

MISCELLANEOUS:

MISC1. PARAMETRIC EQUATIONS: Recognize and give examples of parametric equations. Explain that parametric equations describe more than just a curve in a plane, they describe a curve that is traversed in a particular way. Convert parametric equations to a rectangular equation: what information is lost in the process? Find parametric equations (more than one choice) for a given graph.

MISC2. POLAR COORDINATES: Understand that there is more than one way to describe the position of a point in a plane. Explain polar coordinates. Convert to/from rectangular/polar coordinates. Graph polar equations.

MISC3. Find the PARTIAL FRACTION DECOMPOSITION of a rational function. Do a long division, if necessary, to get the degree of the numerator strictly less than the degree of the denominator. Know the term types that are generated by these factors in the denominator: nonrepeated linear, repeated linear, nonrepeated irreducible quadratic, and repeated irreducible quadratic.

MISC4. VECTORS (two-dimensional): work with vectors both geometrically (as an arrow) and algebraically (as an ordered pair); add, subtract, scale, find the magnitude and direction; solve problems using vectors and right triangle trigonometry

MISC5. ABSOLUTE VALUE: Build on the skills developed in Algebra I and II. Solve sentences of the form $0<|x-c|<\delta$ to prepare students for calculus. Consider sentences like $|x|>-1$ (which are always true) and sentences like $|x|<-1$ (which are always false).