COURSE OBJECTIVES LIST: CALCULUS

Calculus Honors and/or AP Calculus AB are offered. Both courses have the same prerequisites, and cover the same material. Students enrolled in AP Calculus AB have the following additional requirements:

- actual AP problems will be a regular part of homework, quizzes, and tests
- students are required to take the Advanced Placement Test in May
- there is an extra class meeting each week to allow time to explore ideas in greater depth than the normal class schedule allows

PREREQUISITES:

All skills from Algebra I, Geometry, Algebra II, and Precalculus are assumed. A prerequisites test is given during the first week of class to assess knowledge of these prerequisite skills and to locate deficiencies.

In particular, the **Advanced Placement Program Course Description** for MAY 2002–MAY 2003 lists the following prerequisites:

Before studying calculus, all students should complete four years of secondary mathematics designed for college-bound students: courses in which they study algebra, geometry, trigonometry, analytic geometry, and elementary functions. These functions include those that are linear, polynomial, rational, exponential, logarithmic, trigonometric, inverse trigonometric, and piecewise defined. In particular, before studying calculus, students must be familiar with the properties of functions, the algebra of functions, and the graphs of functions. Students must also understand the language of functions (domain and range, odd and even, periodic, symmetry, zeros, intercepts, and so on) and know the values of the trigonometric functions of numbers such as 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$.

The course objectives are elaborated as follows. The order in which the objectives are listed is not necessarily the order in which they will be taught. The following information is taken from the **Advanced Placement Program Course Description** for MAY 2002–MAY 2003.

PHILOSOPHY:

Calculus AB is primarily concerned with developing the students' understanding of the concepts of calculus and providing experience with its methods and applications. The course emphasizes a multirepresentational approach to calculus, with concepts, results, and problems being expressed geometrically, numerically, analytically, and verbally. The connections among these representations also are important.

Broad concepts and widely applicable methods are emphasized. The focus of the course is neither manipulation nor memorization of an extensive taxonomy of functions, curves, theorems, or problem types. Thus, although facility with manipulation and computational competence are important outcomes, they are not the core of the course.

Technology should be used regularly to reinforce the relationships among the multiple representations of functions, to confirm written work, to implement experimentation, and to assist in interpreting results.

Through the use of the unifying themes of derivatives, integrals, limits, approximation, and applications and modeling, the course becomes a cohesive whole rather than a collection of unrelated topics. These themes are developed using all the functions listed in the prerequisites.

GOALS:

GOAL1. Students should be able to work with functions represented in a variety of ways: graphical, numerical, analytical, or verbal. They should understand the connections among these representations.

GOAL2. Students should understand the meaning of the derivative in terms of a rate of change and local linear approximation and should be able to use derivatives to solve a variety of problems.

GOAL3. Students should understand the meaning of the definite integral both as a limit of Riemann sums and as the net accumulation of a rate of change and should be able to use integrals to solve a variety of problems.

GOAL4. Students should understand the relationship between the derivative and the definite integral as expressed in both parts of the Fundamental Theorem of Calculus.

GOAL5. Students should be able to communicate mathematics both orally and in well-written sentences and should be able to explain solutions to problems.

GOAL6. Students should be able to model a written description of a physical situation with a function, a differential equation, or an integral.

GOAL7. Students should be able to use technology to help solve problems, experiment, interpret results, and verify conclusions.

GOAL8. Students should be able to determine the reasonableness of solutions, including sign, size, relative accuracy, and units of measurement.

GOAL9. Students should develop an appreciation of calculus as a coherent body of knowledge and as a human accomplishment.

I. FUNCTIONS, GRAPHS, and LIMITS

Analysis of Graphs.

FGRL1. With the aid of technology, graphs of functions are often easy to produce. The emphasis is on the interplay between the geometric and analytic information and on the use of calculus both to predict and to explain the observed local and global behavior of a function.

Limits of functions (including one-sided limits).

- FGRL2. An intuitive understanding of the limiting process.
- FGRL3. Calculating limits using algebra.
- FGRL4. Estimating limits from graphs or tables of data.

Asymptotic and unbounded behavior.

- FGRL5. Understanding asymptotes in terms of graphical behavior.
- FGRL6. Describing asymptotic behavior in terms of limits involving infinity.
- FGRL7. Comparing relative magnitudes of functions and their rates of change. (For example, contrasting exponential growth, polynomial growth, and logarithmic growth.)

Continuity as a property of functions.

- FGRL8. An intuitive understanding of continuity. (Close values of the domain lead to close values of the range.)
- FGRL9. Understanding continuity in terms of limits.
- FGRL10. Geometric understanding of graphs of continuous functions (Intermediate Value Theorem and Extreme Value Theorem).

II. DERIVATIVES

Concept of the derivative.

- DER1. Derivative presented geometrically, numerically, and analytically.
- DER2. Derivative interpreted as an instantaneous rate of change.
- DER3. Derivative defined as the limit of the difference quotient.
- DER4. Relationship between differentiability and continuity.

Derivative at a point.

- DER5. Slope of a curve at a point. Examples are emphasized, including points at which there are vertical tangents and points at which there are no tangents.
- DER6. Tangent line to a curve at a point and local linear approximation.
- DER7. Instantaneous rate of change as the limit of average rate of change.
- DER8. Approximate rate of change from graphs and tables of values.

Derivative as a function.

- DER9. Corresponding characteristics of graphs of f and f'.
- DER10. Relationship between the increasing and decreasing behavior of f and the sign of f'.
- DER11. The Mean Value Theorem and its geometric consequences.
- DER12. Equations involving derivatives. Verbal descriptions are translated into equations involving derivatives and vice versa.

Second derivatives.

DER13. Corresponding characteristics of the graphs of f , f' , and f'' .

DER14. Relationship between the concavity of f and the sign of f'' .

DER15. Points of inflection as places where concavity changes.

Applications of derivatives.

DER16. Analysis of curves, including the notions of monotonicity and concavity.

DER17. Optimization, both absolute (global) and relative (local) extrema.

DER18. Modeling rates of change, including related rates problems.

DER19. Use of implicit differentiation to find the derivative of an inverse function.

DER20. Interpretation of the derivative as a rate of change in varied applied contexts, including velocity, speed, and acceleration.

Computation of derivatives.

DER21. Knowledge of derivatives of basic functions, including power, exponential, logarithmic, trigonometric, and inverse trigonometric functions.

DER22. Basic rules for the derivative of sums, products, and quotients of functions.

DER23. Chain rule and implicit differentiation.

III. Integrals

Interpretations and properties of definite integrals.

INT1. Computation of Riemann sums using left, right, and midpoint evaluation points.

INT2. Definite integral as a limit of Riemann sums over equal subdivisions.

INT3. Definite integral of the rate of change of a quantity over an interval interpreted as the change of the quantity over the interval:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

INT4. Basic properties of definite integrals. (Examples include additivity and linearity.)

Applications of integrals.

Appropriate integrals are used in a variety of applications to model physical, biological, or economic situations. Although only a sampling of applications can be included in any specific course, students should be able to adapt their knowledge and techniques to solve other similar application problems. Whatever applications are chosen, the emphasis is on:

INT5. using the integral of a rate of change to give accumulated change; or

INT6. using the method of setting up an approximating Riemann sum and representing its limit as a definite integral.

To provide a common foundation, specific applications should include finding:

INT7. the area of a region

INT8. the volume of a solid with known cross sections

INT9. the average value of a function

INT10. the distance traveled by a particle along a line

Fundamental Theorem of Calculus.

INT11. Use of the Fundamental Theorem to evaluate definite integrals.

INT12. Use of the Fundamental Theorem to represent a particular antiderivative, and the analytical and graphical analysis of functions so defined.

Techniques of antidifferentiation.

INT13. Antiderivatives following directly from derivatives of basic functions.

INT14. Antiderivatives by substitution of variables (including change of limits for definite integrals).

Applications of antidifferentiation.

INT15. Finding specific antiderivatives using initial conditions, including applications to motions along a line.

INT16. Solving separable differential equations and using them in modeling. In particular, studying the equation $y' = ky$ and exponential growth.

Numerical approximations to definite integrals.

INT17. Use of Riemann and trapezoidal sums to approximate definite integrals of functions represented algebraically, geometrically, and by tables of values.

The following objective will be included in the topic outline for Calculus AB for the 2003–2004 academic year for the 2004 AP Examinations:

INT18. Geometric interpretation of differential equations via slope fields and the relationship between slope fields and solution curves for differential equations.

MUST-KNOW MATERIAL FOR CALCULUS

MISCELLANEOUS:

interval notation: (a, b), $[a, b]$, $(a, b]$, (a, ∞), etc.

Rewrite radicals as fractional exponents: $\sqrt[3]{x} = x^{1/3}$, $\sqrt{x^3} = x^{3/2}$ etc.

An implication 'If A then B' is equivalent to its contrapositive 'If (not B) then (not A)'

To go from graph of $y = f(x)$ to graph of $x = f(y)$: take the (familiar) graph of $y = f(x)$, rotate 90 degrees clockwise, then flip about the horizontal axis.

TEST POINT METHOD: for solving $f(x) > 0$: there are only two types of places where a function can change from positive to negative (or vice versa): where it equals zero, or at a break. Locate all such places, and check the resulting subintervals.

GEOMETRY:

Circle with radius r: AREA $= \pi r^2$, CIRCUMFERENCE $= 2\pi r$, DIAMETER $= 2r$

Sphere with radius r: VOLUME $= \frac{4}{3}\pi r^3$, SURFACE AREA $= 4\pi r^2$

Area of a triangle:

base b and height h, AREA $= \frac{1}{2}bh$

sides a and b with included angle θ: AREA $= \frac{1}{2}ab\sin\theta$

Triangles: angles sum to 180°; longest side is opposite biggest angle, etc.

Consider an arbitrary triangle with angles A, B, C and opposite sides a, b, c:

Law of Sines: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

Law of Cosines: $a^2 = b^2 + c^2 - 2bc\cos A$

Similar triangles: have the same angles; scaling factor in going from one to the other

Right triangles: have a 90° angle; longest side is called the hypotenuse;

the Pythagorean Theorem: $a^2 + b^2 = c^2$

Trapezoid with bases b_1 and b_2 and height h: AREA $= \frac{(b_1+b_2)}{2} \cdot h$ (average the bases and multiply by the height)

cylinder (2 parallel congruent plane figures of area A, perpendicular distance between planes is h): VOLUME $= Ah$

right circular cylinder: VOLUME $= \pi r^2 h$

cone (a plane figure of area A, a point, all lines connecting; h is perpendicular distance from point to plane): VOLUME $= \frac{1}{3}Ah$

right circular cone: VOLUME $= \frac{1}{3}\pi r^2 h$

TRIGONOMETRY:

RADIAN MEASURE: the radian measure of an angle is the length of the arc on the unit circle: positive is counterclockwise.

Right triangle definitions: SOHCAHTOA

Unit circle definitions; lay off angle x

$\sin x$ is the y-value of the point

$\cos x$ is the x-value of the point

$\tan x = \frac{\sin x}{\cos x}$

$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

$\sec x = \frac{1}{\cos x}$

$\csc x = \frac{1}{\sin x}$

$\sin^{-1} x = \arcsin x$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x.

$\cos^{-1} x = \arccos x$ is the angle between 0 and π whose cosine is x.

$\tan^{-1} x = \arctan x$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose tangent is x.

Note: $\sin^2 x$ means $\big(\sin x\big)^2$ etc.

Double-angle formulas: $\sin 2x = 2 \sin x \cos x$; $\cos 2x = \cos^2 x - \sin^2 x$

the Pythagorean Identity: $\sin^2 x + \cos^2 x = 1$

Two special triangles: 30°-60°-90° and 45°-45°-90°

FUNCTIONS:

$f(x)$ is the output from the function f when the input is x.

Functions have the property that each input has exactly one corresponding output.

ZERO of a function: an input whose output is zero

DOMAIN of a function: set of allowable inputs

RANGE of a function: its output set

GRAPH of a function: the picture of its (input,output) pairs

GRAPHS of BASIC MODELS:

constant $(y = k)$

$y = x^2$ and higher powers

$y = x^3$ and higher powers

$y = \frac{1}{x}$

$y = \sqrt{x}$

$y = |x|$

$y = \mathrm{e}^x$

$y = \ln x$

$y = \sin x$

$y = \cos x$

$y = \tan x$

$y = \sec x$

$y = [[x]]$, the greatest integer function, $[[x]]$ is the greatest integer less than or equal to x

COMPOSITIONS OF FUNCTIONS: $f(g(x))$ means g acts first, f acts last

EVEN function: $f(-x) = f(x)$; when inputs are opposites, outputs are the same

ODD function: $f(-x) = -f(x)$; when inputs are opposites, outputs are opposites

ONE-TO-ONE FUNCTION: Each output has exactly one corresponding input; graph passes both a horizontal and vertical line test; the inputs and outputs can be tied together with strings

INVERSE FUNCTIONS:

If f is 1-1, then its inverse f^{-1} 'undoes' what f did: $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

the domains and ranges of f and f^{-1} are switched

the graphs of f and f^{-1} are reflections about the line $y = x$

if (a, b) is on the graph of f, then (b, a) is on the graph of f^{-1}

TRANSFORMATIONS of functions: start with $y = f(x)$

working with y is intuitive:

$y = f(x) + 3$ moves up 3

$y = 3f(x)$ multiplies all y-values by 3 (vertical stretch)

$y = -f(x)$ multiplies the y-values by -1; reflects about the x-axis

working with x is counter-intuitive:

$y = f(x - 3)$; replace every x with $x - 3$; moves to the RIGHT 3

$y = f(3x)$; replace every x with $3x$; $(a, b) \mapsto (\frac{a}{3}, b)$; horizontal compression

$y = f(-x)$; replace every x with $-x$; reflects about the y-axis

LINES:

linear functions: $y = mx + b$ or $ax + by + c = 0$; equal changes in x give rise to equal changes in y

slope: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$; if $m = 3$, then the y-values are changing 3 times as fast as the x-values

point-slope form: $y - y_1 = m(x - x_1)$

parallel lines have the same slope; perpendicular lines have slopes that are opposite reciprocals

horizontal lines: $y = c$; have zero slope   vertical lines: $x = c$; have no slope

QUADRATIC FUNCTIONS:

$f(x) = ax^2 + bx + c$; graph as parabolas; $a > 0$ holds water, $a < 0$ sheds water

vertex: at $x = -\frac{b}{2a}$

POLYNOMIALS:

Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.

degree of P: highest exponent

As $x \to \pm\infty$, a polynomial 'looks like' its highest power term.

The following are equivalent:

- c is a zero of P
- $x - c$ is a factor of $P(x)$
- $P(c) = 0$
- the point $(c, 0)$ is on the graph of P
- the graph of P crosses the x-axis at c
- $x - c$ goes into $P(x)$ evenly (remainder $= 0$)

EXPONENTIAL FUNCTIONS: $y = b^t$

Allowable bases: $b > 0$, $b \neq 1$

Increasing when $b > 1$; decreasing when $0 < b < 1$

Common form: $A(t) = A_0 \mathrm{e}^{kt}$; A_0 is the amount at time 0

Every exponential function can be written with ANY allowable base, so use whatever base is convenient.

For equal changes in x, y gets MULTIPLIED by a constant (that depends both on the base of the exponential function, and the change in x)

Doubling time: for an increasing exponential function, it always takes the same amount of time for a quantity to double

half-life: for a decreasing exponential function, it always takes the same amount of time for a quantity to be cut in half

LOGARITHMS: $y = \log_b x$

Allowable bases: $b > 0$, $b \neq 1$

Increasing when $b > 1$; decreasing when $0 < b < 1$

Laws work for all allowable bases:

$\ln x = \log_{\mathrm{e}} x$ is the natural logarithm

$\ln xy = \ln x + \ln y$ (the log of a product is the sum of the logs)

$\ln \frac{x}{y} = \ln x - \ln y$ (the log of a quotient is the difference of the logs)

$\ln x^y = y \ln x$ (you can bring powers down)

change-of-base formula: $\log_b x = \dfrac{\log_a x}{\log_a b}$

$y = \mathrm{e}^x$ and $y = \ln x$ are inverse functions; use this idea to solve exponential and logarithmic equations

A log is an exponent! '$\log_3 5$' is the POWER that 3 must be raised to, to get 5

EXPONENTIAL FUNCTIONS grow faster than POWER FUNCTIONS grow faster than LOGARITHMIC FUNCTIONS

ABSOLUTE VALUE:

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

For $c > 0$,

$|x| < c \iff -c < x < c$

$|x| > c \iff x > c$ or $x < -c$

$|x| = c \iff x = \pm c$

$0 < |x - c| < \delta \iff x \in (c - \delta, c) \cup (c, c + \delta)$ (punctured neighborhood about c)

LIMITS:

Consider the limit statement: $\lim\limits_{x \to c} f(x) = \ell$

low-level understanding: when x is close to c, $f(x)$ is close to ℓ

higher level: we can make the values of $f(x)$ as close to ℓ as we like, by taking x to be sufficiently close to c, but not equal to c

Precisely: $\forall\, \epsilon > 0 \ \exists\, \delta > 0$ s.t. if $0 < |x - c| < \delta$ then $|f(x) - \ell| < \epsilon$

When we evaluate a limit as $x \to c$, we never let x equal c

$x \to c^+$ means x approaches c from the right-hand side

$x \to c^-$ means x approaches c from the left-hand side

LIMIT LAWS: Work nicely! Providing the individual limits exist, the limit of a sum is the sum of the limits (same for difference, products, quotients, etc.)

If you have a continuous function (see below) then evaluating a limit is as easy as DIRECT SUBSTITUTION.

BE CAREFUL! If you're working with a discontinuous function (e.g., greatest integer function, some piecewise-defined functions), then direct substitution MAY NOT WORK. Try l'Hospital's rule, renaming, graphing, etc.

An important limit: $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

CONTINUITY:

Low-level understanding: no breaks in the graph

higher level: when inputs are close, outputs are close

The following are equivalent:

- f is continuous at c
- $\lim_{x \to c} f(x) = f(c)$
  (when f is continuous at c, then evaluating the limit is as easy as direct substitution)
- $\lim_{h \to 0} f(c+h) = f(c)$
- As $x \to c$, $f(x) \to f(c)$

INTERMEDIATE VALUE THEOREM:

Suppose f in continuous on $[a, b]$, and N is a number between $f(a)$ and $f(b)$. Then there exists a number c between a and b for which $f(c) = N$.

(If a graph has no breaks, and you travel along the graph from one point to another, you must pass through ALL the y-values in between; i.e., all the intermediate values.)

EXTREME VALUE (MAX/MIN) THEOREM:

Let f be continuous on a closed interval $[a, b]$. Then f attains both an absolute maximum value $f(c)$ and absolute minimum value $f(d)$ for some c and d in $[a, b]$.

(This theorem GUARANTEES the existence of a 'highest' and 'lowest' point on a graph under appropriate conditions.)

MEAN VALUE THEOREM:

Let f be differentiable on $[a, b]$. Then there exists a number c between a and b for which $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

(This theorem guarantees a place where the instantaneous rate of change is the same as the average rate of change under appropriate conditions.)

DERIVATIVES:

The following are equivalent:

- $f'(c) = m$
- $\lim_{h\to 0} \dfrac{f(c+h) - f(c)}{h} = m$
- The slope of the tangent line to the graph of f at the point $(c, f(c))$ is m
- f is differentiable at c (and the value of the derivative is m)
- $\lim_{x\to c} \dfrac{f(x) - f(c)}{x - c} = m$
- The instantaneous rate of change of f at $(c, f(c))$ is m
- When $x = c$, the function values are changing m times as fast as the inputs

Suppose $f'(2) = 5$. Roughly: when x changes by 1 (from 2 to 3), we expect y to go up by ABOUT 5. Or, when x changes by -1 (from 2 to 1), we expect y to go down by ABOUT 5.

The UNITS of $f'(c)$ are the units of $f(x)$ (the outputs from f) divided by the units of x (the inputs to f)

If a function is differentiable, then its graph is SMOOTH: it has non-vertical tangent lines everywhere.

A function is NON-DIFFERENTIABLE at: vertical tangent lines; kinks; discontinuities

Theorem: If f is differentiable at x, then f is continuous at x

Contrapositive: If f is not continuous at x, then f is not differentiable at x

Leibniz notation versus prime notation: $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$ etc.

Linear Approximation (linearization): a function is best approximated at a point by its tangent line: at $(c, f(c))$ we have: $L(x) = f(c) + f'(c)(x - c) \approx f(x)$

If $f'(x) > 0$ then f is increasing

If $f'(x) < 0$ then f is decreasing

If the SLOPES are INCREASING (f' increasing; $f'' > 0$) then f is concave up

If the SLOPES are DECREASING (f' decreasing; $f'' < 0$) then f is concave down

Remember: a function can increase in basically three different ways: linearly, concave up; concave down

Inflection point: where the concavity changes (from concave up to down, or down to up): candidates are where $f''(c) = 0$ or $f''(c)$ does not exist.

LOCAL MAX/MIN:

A local max/min for a function can only occur at three types of places (called the 'critical points'):

- where $f'(c) = 0$
- where $f'(c)$ does not exist
- at ENDPOINTS of domain of f

So, to find max/min, locate all candidates, and check them.

Careful: a critical point is not necessarily a max or min!

FIRST DERIVATIVE TEST: Check signs of first derivative to the left/right of a candidate (where the function is continuous) to decide if it is a max or min.

Why is continuity needed? See the sketch below—the test would tell us that there's a local max at c!

SECOND DERIVATIVE TEST: If concave up at a candidate; it's a min. If concave down at a candidate, it's a max.

AVERAGE RATE OF CHANGE: The average rate of change of f on the interval $[a, b]$ is $\dfrac{f(b) - f(a)}{b - a}$; this is the slope of the line between $(a, f(a))$ and $(b, f(b))$

DIFFERENTIATION FORMULAS:

Be able to GENERALIZE all these formulas: replace x by $f(x)$, and multiply by $f'(x)$

$\frac{d}{dx} x^n = nx^{n-1}$ generalize: $\frac{d}{dx}\big(f(x)\big)^n = n\big(f(x)\big)^{n-1} \cdot f'(x)$

$\frac{d}{dx} cf(x) = c\frac{d}{dx} f(x)$ (you can slide constants out)

the derivative of a sum/difference is the sum/difference of the derivatives

$\frac{d}{dx} \mathrm{e}^x = \mathrm{e}^x$ (the y-value of the point tells you how fast the function is changing at that point)

PRODUCT RULE: $\frac{d}{dx} f(x)g(x) = f(x)g'(x) + g(x)f'(x)$ (the derivative of a product is NOT!! NOT!! NOT!! the product of the derivatives)

QUOTIENT RULE: $\dfrac{d}{dx}\dfrac{f(x)}{g(x)} = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$ (the derivative of a quotient is NOT!! NOT!! NOT!! the quotient of the derivatives)

x MUST BE MEASURED IN RADIANS FOR THESE FORMULAS TO BE CORRECT:

$\frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx} \cos x = -\sin x$

$\frac{d}{dx} \tan x = \sec^2 x$

$\frac{d}{dx} \cot x = -\csc^2 x$

$\frac{d}{dx} \sec x = \sec x \tan x$

$\frac{d}{dx} \csc x = -\csc x \cot x$

Chain Rule: $\frac{d}{dx} f(g(x)) = f'(g(x) \cdot g'(x)$; how to differentiate composite functions

$\frac{d}{dx} a^x = a^x \ln a$

$$\frac{d}{dx} \arcsin x = \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

$\frac{d}{dx} \arccos x = \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx} \arctan x = \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

$\frac{d}{dx} \ln x = \frac{1}{x}$

$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$

derivative of an inverse function: $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

So if (a, b) is on the graph of f with slope of tangent line m,

then (b, a) is on the graph of f^{-1} with slope of tangent line $\frac{1}{m}$!

IMPLICIT DIFFERENTIATION:

Whenever you see y, treat it as a function of x and differentiate accordingly.

For example: $\frac{d}{dx} y^2 = 2y \frac{dy}{dx}$

For example: $\frac{d}{dx} xy = x\frac{dy}{dx} + y$

LOGARITHMIC DIFFERENTIATION:

Use this to differentiate complicated products or quotients; also to differentiate variable stuff raised to a variable power. First take logs, then differentiate!

PARTICLE MOVING ON A NUMBER LINE:

Let $s(t)$ denote the position at time t.

Then, $s'(t) = v(t)$ is the velocity; positive is moving to the right; negative to the left.

$s''(t) = v'(t) = a(t)$ is the acceleration.

'Speeding up' means moving to the right faster and faster ($v(t) > 0$ and $a(t) > 0$) or moving to the left faster and faster ($v(t) < 0$ and $a(t) < 0$). Thus, the particle speeds up when velocity and acceleration have the same sign (both positive, or both negative).

Note: speed $= |v(t)|$

Suppose you're given the velocity of a particle traveling along a number line, $v(t)$. Then, total distance traveled from t_1 to t_2 is given by $\int_{t_1}^{t_2} |v(t)|\, dt$; i.e., integrate the speed.

However the total DISPLACEMENT is $\int_{t_1}^{t_2} v(t)\, dt = s(t_2) - s(t_1)$. Notice that if you start at 0, move to the right 5 and then to the left 5, your total displacement is 0 but the total distance traveled is 10.

RELATED RATE PROBLEMS:

Ask: What is changing with time? Rates are derivatives! Write down SOMETHING THAT IS TRUE that involves what you're interested in. (Look for: similar figures, right triangles, etc.)

Remember: if x is changing with time, then the derivative of x^2 is $2x\frac{dx}{dt}$.

OPTIMIZATION PROBLEMS:

Find the CANDIDATES for local max/min: endpoints, places where the derivative is zero, places where the derivative doesn't exist.

Use the First Derivative Test or Second Derivative Test to check whether they're a max or min.

If you want an ABSOLUTE max/min, find ALL the local max/min, and choose the highest/lowest from these.

Remember, you CAN'T USE YOUR CALCULATOR to locate max/mins; this is NOT an allowable operation!

ANTIDERIVATIVES:

A function $F(x)$ is an ANTIDERIVATIVE of $f(x)$ if and only if $F'(x) = f(x)$; i.e., F is a function whose derivative is f. Antiderivatives 'undo' derivatives. An antiderivative has a specified derivative, and this derivative determines the shape, but not the vertical translation. So, if you have ONE antiderivative, then you have an infinite number—they all differ by a constant.

The symbol $\int f(x)\,dx$ denotes all the antiderivatives of $f(x)$.

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ for $n \neq -1$

$\int \frac{1}{x} = \ln|x| + C$

$\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = \sin x + C$

$\int \mathrm{e}^{kx}\,dx = \frac{1}{k}\mathrm{e}^{kx} + C$

$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$

Any CONTINUOUS function f has an antiderivative: the function $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$. That is, the function that finds AREA under the graph of f is an ANTIDERIVATIVE of f!

DEFINITE INTEGRALS:

the definite integral of f from a to b is denoted by $\int_a^b f(x)\,dx$ and is defined as follows:

Divide $[a,b]$ into n equal subintervals, each of length $\Delta x = \frac{b-a}{n}$. Choose x_i^* from the i^{th} subinterval. Then,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

You can approximate definite integrals with rectangles (left-hand; right-hand; midpoint), with trapezoids, even with parabolas.

The definite integral gives information about the (signed) area trapped between the graph of f and the x-axis: area above is treated as positive; area below is negative.

Caution: if $\int_a^b f(x)\,dx = 0$, this only means that there is the same amount of area ABOVE the x-axis as BELOW on the interval $[a, b]$.

When you have a definite integral problem that can be solved with simple geometry formulas (triangles, trapezoids, circles) then USE GEOMETRY to find the definite integral—it's much more efficient!

EVALUATION THEOREM: If F is any antiderivative of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

TOTAL CHANGE THEOREM: When you integrate a rate of change, you get total change:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Rewrite this as

$$f(b) = f(a) + \int_a^b f'(x)\,dx$$

and think of it like this: If you want to know the value of f at b, first find the value of f at someplace you know (a), and then see how much f has CHANGED BY in going from a to b ($\int_a^b f'(x)\,dx$).

SUBSTITUTION METHOD for antidifferentiation/integration: Choose u to be something whose derivative is in the integrand, perhaps off by a constant. Often, u is something in parentheses, the argument of a function, something in an exponent, etc. Be sure to change the limits if you have a definite integral.

APPLICATIONS OF INTEGRATION:

AREAS BETWEEN CURVES: Find the intersection points, write the area of a typical 'slice' and 'sum' appropriately.

vertical slices: AREA $= \int_a^b \big(f(x) - g(x)\big)\,dx$

horizontal slices: AREA $= \int_c^d \big(f(y) - g(y)\big)\,dy$ (Will need to solve for x in terms of y)

VOLUMES OF REVOLUTION:

DISK METHOD: revolve $y = f(x)$ about the x-axis on $[a, b]$; volume of the resulting solid is $\int_a^b \pi\big(f(x)\big)^2\,dx$.

SHELL METHOD: revolve $y = f(x)$ on $[a, b]$ about the y-axis; volume of the resulting solid is $\int_a^b 2\pi x f(x)\,dx$.

AVERAGE VALUE OF A FUNCTION: the average value of $f(x)$ on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$; you're 'summing up' the outputs from a to b (the integral), and then 'dividing by how many you have' (the length of the interval). If you 'smush' the area into rectangular shape, the average value gives the height of the rectangle.

Caution: Don't mix up 'average rate of change' and 'average value'!

CONNECTION BETWEEN **average value** and the **average rate of change**:

$\dfrac{1}{b-a}\displaystyle\int_a^b f'(x)\,dx = \dfrac{f(b) - f(a)}{b-a}$: 'averaging' the values of $f'(x)$ on $[a, b]$ gives the average rate of change of f on $[a, b]$

SEPARABLE DIFFERENTIAL EQUATIONS:

Get all the y's on one side, and all the x's on the other side. Integrate. Don't forget the constant of integration. Use a given condition to solve for this constant.

SLOPE FIELDS:

Slope fields help us to visualize the solutions to first-order differential equations. Get a formula for the derivative in terms of x and y; find the slope at many different points. The resulting 'field of slopes' helps us to see the shapes of the solution curves.