ALGEBRA II OBJECTIVE: GR5

vertical scaling (stretching/shrinking): going from $y = f(x)$ to $y = kf(x)$ for $k > 0$

horizontal scaling (stretching/shrinking): going from $y = f(x)$ to $y = f(kx)$ for $k > 0$

DISCUSSION OF CONCEPT:

There are things that you can DO to an equation of the form $y = f(x)$ that will change the graph in a variety of ways. For example, you can move the graph up or down, left or right, reflect about the x or y axes, stretch or shrink vertically or horizontally.

An understanding of these transformations makes it easy to graph a wide variety of functions, by starting with a 'basic model' and then applying a sequence of transformations to change it to the desired function.

In this discussion, we will explore stretching and shrinking a graph both vertically and horizontally.

When you finish studying this objective, you should be able to do a problem like this:

GRAPH $y = 2e^{5x}$:

- Start with the graph of $y = e^x$. (This is the 'basic model'.)
- Multiply the previous y-values by 2 , giving the new equation $y = 2e^x$.

  This produces a vertical stretch, where the y-values on the graph get multiplied by 2 .
- Replace every x by $5x$, giving the new equation $y = 2e^{5x}$.

  This produces a horizontal shrink, where the x-values on the graph get divided by 5 .

Here are ideas that are needed to understand graphical transformations.
First, some ideas regarding functions and the graph of a function:

- A function is a rule: it takes an input, and gives a unique output.
- If x is the input to a function f , then the unique output is called $f(x)$ (which is read as 'f of x').
- The *graph* of a function is a picture of *all* of its (input, output) pairs. We put the inputs along the horizontal axis (the x-axis), and the outputs along the vertical axis (the y-axis).
- Thus, the graph of a function f is a picture of all points of the form $(x, \overbrace{f(x)}^{y\text{-value}})$. Here, x is the input, and $f(x)$ is the corresponding output.
- The equation $y = f(x)$ is an equation in two variables, x and y . A solution is a choice for x and a choice for y that makes the equation true. Of course, in order for this equation to be true, y must equal $f(x)$.

  Thus, solutions to the equation $y = f(x)$ are points of the form $(x, \overbrace{f(x)}^{y\text{-value}})$.
- Compare the previous two ideas! You see that the requests 'graph the function f' and 'graph the equation $y = f(x)$' mean exactly the same thing.

  To "graph the function f" means to show all points of the form $(x, f(x))$.

  To "graph the equation $y = f(x)$" means to show all points of the form $(x, f(x))$.

Ideas regarding vertical scaling (stretching/shrinking):

- Points on the graph of $y = f(x)$ are of the form $(x, f(x))$.

  Points on the graph of $y = 3f(x)$ are of the form $(x, 3f(x))$.

  Thus, the graph of $y = 3f(x)$ is found by taking the graph of $y = f(x)$ and multiplying the y-values by 3. This moves the points farther from the x-axis, which makes the graph steeper.

- Points on the graph of $y = f(x)$ are of the form $(x, f(x))$.

  Points on the graph of $y = \frac{1}{3}f(x)$ are of the form $(x, \frac{1}{3}f(x))$.

  Thus, the graph of $y = \frac{1}{3}f(x)$ is found by taking the graph of $y = f(x)$ and multiplying the y-values by $\frac{1}{3}$. This moves the points closer to the x-axis, which makes the graph flatter.

- **Transformations involving y work the way you would expect them to work—they are intuitive.**

- Here is the thought process you should use when you are given the graph of $y = f(x)$ and asked about the graph of $y = 3f(x)$:

  original equation: $y = f(x)$

  new equation: $$\overbrace{y}^{\text{the new } y\text{-values}} \quad \overbrace{=}^{\text{are}} \quad \overbrace{3}^{\text{three times}} \quad \overbrace{f(x)}^{\text{the previous } y\text{-values}}$$

- Summary of vertical scaling:

  Let $k > 1$.

  Start with the equation $y = f(x)$.

  Multiply the previous y-values by k, giving the new equation $y = kf(x)$.

  The y-values are being multiplied by a number greater than 1, so they move farther from the x-axis. This makes the graph steeper, and is called a vertical stretch.

  Let $0 < k < 1$.

  Start with the equation $y = f(x)$.

  Multiply the previous y-values by k, giving the new equation $y = kf(x)$.

  The y-values are being multiplied by a number between 0 and 1, so they move closer to the x-axis. This makes the graph flatter, and is called a vertical shrink.

  In both cases, a point (a, b) on the graph of $y = f(x)$ moves to a point (a, kb) on the graph of $y = kf(x)$.

  This transformation type is formally called *vertical scaling (stretching/shrinking)*.

Ideas regarding horizontal scaling (stretching/shrinking):

- Points on the graph of $y = f(x)$ are of the form $(x, f(x))$.

  Points on the graph of $y = f(3x)$ are of the form $(x, f(3x))$.

  How can we locate these desired points $(x, f(3x))$?

  First, go to the point $(3x, f(3x))$ on the graph of $y = f(x)$.

  This point has the y-value that we want, but it has the wrong x-value.

  The x-value of this point is $3x$, but the desired x-value is just x. Thus, the current x-value must be divided by 3; the y-value remains the same. This gives the desired point $(x, f(3x))$.

  Thus, the graph of $y = f(3x)$ is the same as the graph of $y = f(x)$, except that the x-values have been divided by 3 (NOT multiplied by 3, which you might expect). Notice that dividing the x-values by 3 moves them closer to the y-axis.

- **Transformations involving x do NOT work the way you would expect them to work—they are counter-intuitive—they are against your intuition.**

- Here is the thought process you should use when you are given the graph of $y = f(x)$ and asked about the graph of $y = f(3x)$:

  original equation: $y = f(x)$

  new equation: $y = f(\overbrace{\quad 3x \quad}^{\text{replace } x \text{ by } 3x})$

  Replacing every x by $3x$ in an equation causes the x-values on the graph to be DIVIDED by 3.

- Summary of horizontal scaling:

  Let $k > 1$.

  Start with the equation $y = f(x)$.

  Replace every x by kx, giving the new equation $y = f(kx)$.

  This causes the x-values on the graph to be DIVIDED by k, which moves the points closer to the y-axis. This is called a horizontal shrink.

  A point (a, b) on the graph of $y = f(x)$ moves to a point $(\frac{a}{k}, b)$ on the graph of $y = f(kx)$.

  Let $k > 1$.

  Start with the equation $y = f(x)$.

  Replace every x by $\frac{x}{k}$, giving the new equation $y = f(\frac{x}{k})$.

  This causes the x-values on the graph to be MULTIPLIED by k, which moves the points farther away from the y-axis. This is called a horizontal stretch.

  A point (a, b) on the graph of $y = f(x)$ moves to a point (ka, b) on the graph of $y = f(\frac{x}{k})$.

  This transformation type is formally called *horizontal scaling (stretching/shrinking).*

Notice that **different words** are used when talking about transformations involving y, and transformations involving x.

For transformations involving y (that is, transformations that change the y-values of the points), we say:

DO THIS to the previous y-value.

For transformations involving x (that is, transformations that change the x-values of the points), we say:

REPLACE the previous x-values by ....

Here are some examples that combine horizontal and vertical translations and scaling.

EXAMPLE:

State the transformations that take the graph of $y = f(x)$ to the graph of $y = 5f(3x-1)+4$.

Equation	Action	Graphical Result
$y = f(x)$	(starting place)	
$y = f(x-1)$	replace every x by $x-1$	move RIGHT 1
$y = f(3x-1)$	replace every x by $3x$	horizontal shrink, $(a,b) \mapsto (\frac{a}{3}, b)$
$y = 5f(3x-1)$	multiply the previous y-values by 5	vertical stretch, $(a,b) \mapsto (a, 5b)$
$y = 5f(3x-1)+4$	add 4 to the previous y-values	move UP 4

EXAMPLE:

State the transformations that take the graph of $y = \sqrt{x}$ to the graph of $y = 3\sqrt{\frac{x}{5}+2} - 1$.

Equation	Action	Graphical Result
$y = \sqrt{x}$	(starting place)	
$y = 3\sqrt{x}$	multiply the previous y-values by 3	vertical stretch, $(a,b) \mapsto (a, 3b)$
$y = 3\sqrt{x+2}$	replace every x by $x+2$	move LEFT 2
$y = 3\sqrt{\frac{x}{5}+2}$	replace every x by $\frac{x}{5}$	horizontal stretch, $(a,b) \mapsto (5a, b)$
$y = 3\sqrt{\frac{x}{5}+2} - 1$	subtract 1 from the previous y-values	move DOWN 1