

ALGEBRA II OBJECTIVE: GR5

vertical scaling (stretching/shrinking): going from $y = f(x)$ to $y = kf(x)$ for $k > 0$

horizontal scaling (stretching/shrinking): going from $y = f(x)$ to $y = f(kx)$ for $k > 0$

DISCUSSION OF CONCEPT:

There are things that you can DO to an equation of the form $y = f(x)$ that will change the graph in a variety of ways. For example, you can move the graph up or down, left or right, reflect about the x or y axes, stretch or shrink vertically or horizontally.

An understanding of these transformations makes it easy to graph a wide variety of functions, by starting with a ‘basic model’ and then applying a sequence of transformations to change it to the desired function.

In this discussion, we will explore stretching and shrinking a graph both vertically and horizontally.

When you finish studying this objective, you should be able to do a problem like this:

GRAPH $y = 2e^{5x}$:

- Start with the graph of $y = e^x$. (This is the ‘basic model’.)
- Multiply the previous y -values by 2, giving the new equation $y = 2e^x$.
This produces a vertical stretch, where the y -values on the graph get multiplied by 2.
- Replace every x by $5x$, giving the new equation $y = 2e^{5x}$.
This produces a horizontal shrink, where the x -values on the graph get divided by 5.

Here are ideas that are needed to understand graphical transformations.

First, some ideas regarding functions and the graph of a function:

- A function is a rule: it takes an input, and gives a unique output.
- If x is the input to a function f , then the unique output is called $f(x)$ (which is read as ‘ f of x ’).
- The *graph* of a function is a picture of *all* of its (input, output) pairs. We put the inputs along the horizontal axis (the x -axis), and the outputs along the vertical axis (the y -axis).
- Thus, the graph of a function f is a picture of all points of the form $(x, \overbrace{f(x)}^{y\text{-value}})$. Here, x is the input, and $f(x)$ is the corresponding output.
- The equation $y = f(x)$ is an equation in two variables, x and y . A solution is a choice for x and a choice for y that makes the equation true. Of course, in order for this equation to be true, y must equal $f(x)$.

Thus, solutions to the equation $y = f(x)$ are points of the form $(x, \overbrace{f(x)}^{y\text{-value}})$.

- Compare the previous two ideas! You see that the requests ‘graph the function f ’ and ‘graph the equation $y = f(x)$ ’ mean exactly the same thing.

To “graph the function f ” means to show all points of the form $(x, f(x))$.

To “graph the equation $y = f(x)$ ” means to show all points of the form $(x, f(x))$.

Ideas regarding vertical scaling (stretching/shrinking):

- Points on the graph of $y = f(x)$ are of the form $(x, f(x))$.
Points on the graph of $y = 3f(x)$ are of the form $(x, 3f(x))$.
Thus, the graph of $y = 3f(x)$ is found by taking the graph of $y = f(x)$ and multiplying the y -values by 3. This moves the points farther from the x -axis, which makes the graph steeper.
- Points on the graph of $y = f(x)$ are of the form $(x, f(x))$.
Points on the graph of $y = \frac{1}{3}f(x)$ are of the form $(x, \frac{1}{3}f(x))$.
Thus, the graph of $y = \frac{1}{3}f(x)$ is found by taking the graph of $y = f(x)$ and multiplying the y -values by $\frac{1}{3}$. This moves the points closer to the x -axis, which makes the graph flatter.
- **Transformations involving y work the way you would expect them to work—they are intuitive.**
- Here is the thought process you should use when you are given the graph of $y = f(x)$ and asked about the graph of $y = 3f(x)$:

original equation:

$$y = f(x)$$

new equation:

$$\underbrace{\text{the new } y\text{-values}}_y \text{ are } \underbrace{=}_{= 3} \underbrace{\text{three times}}_3 \underbrace{\text{the previous } y\text{-values}}_{f(x)}$$

- Summary of vertical scaling:
Let $k > 1$.
Start with the equation $y = f(x)$.
Multiply the previous y -values by k , giving the new equation $y = kf(x)$.
The y -values are being multiplied by a number greater than 1, so they move farther from the x -axis. This makes the graph steeper, and is called a vertical stretch.
Let $0 < k < 1$.
Start with the equation $y = f(x)$.
Multiply the previous y -values by k , giving the new equation $y = kf(x)$.
The y -values are being multiplied by a number between 0 and 1, so they move closer to the x -axis. This makes the graph flatter, and is called a vertical shrink.
In both cases, a point (a, b) on the graph of $y = f(x)$ moves to a point (a, kb) on the graph of $y = kf(x)$.
This transformation type is formally called *vertical scaling (stretching/shrinking)*.

Ideas regarding horizontal scaling (stretching/shrinking):

- Points on the graph of $y = f(x)$ are of the form $(x, f(x))$.
Points on the graph of $y = f(3x)$ are of the form $(x, f(3x))$.
How can we locate these desired points $(x, f(3x))$?
First, go to the point $(3x, f(3x))$ on the graph of $y = f(x)$.
This point has the y -value that we want, but it has the wrong x -value.
The x -value of this point is $3x$, but the desired x -value is just x . Thus, the current x -value must be divided by 3; the y -value remains the same. This gives the desired point $(x, f(3x))$.
Thus, the graph of $y = f(3x)$ is the same as the graph of $y = f(x)$, except that the x -values have been divided by 3 (NOT multiplied by 3, which you might expect). Notice that dividing the x -values by 3 moves them closer to the y -axis.
- **Transformations involving x do NOT work the way you would expect them to work—they are counter-intuitive—they are against your intuition.**
- Here is the thought process you should use when you are given the graph of $y = f(x)$ and asked about the graph of $y = f(3x)$:

original equation:

$$y = f(x)$$

new equation:

$$y = f\left(\overbrace{3x}^{\text{replace } x \text{ by } 3x}\right)$$

Replacing every x by $3x$ in an equation causes the x -values on the graph to be DIVIDED by 3.

- Summary of horizontal scaling:
Let $k > 1$.
Start with the equation $y = f(x)$.
Replace every x by kx , giving the new equation $y = f(kx)$.
This causes the x -values on the graph to be DIVIDED by k , which moves the points closer to the y -axis. This is called a horizontal shrink.
A point (a, b) on the graph of $y = f(x)$ moves to a point $(\frac{a}{k}, b)$ on the graph of $y = f(kx)$.
Let $k > 1$.
Start with the equation $y = f(x)$.
Replace every x by $\frac{x}{k}$, giving the new equation $y = f(\frac{x}{k})$.
This causes the x -values on the graph to be MULTIPLIED by k , which moves the points farther away from the y -axis. This is called a horizontal stretch.
A point (a, b) on the graph of $y = f(x)$ moves to a point (ka, b) on the graph of $y = f(\frac{x}{k})$.
This transformation type is formally called *horizontal scaling (stretching/shrinking)*.

Notice that **different words** are used when talking about transformations involving y , and transformations involving x .

For transformations involving y (that is, transformations that change the y -values of the points), we say:

DO THIS to the previous y -value.

For transformations involving x (that is, transformations that change the x -values of the points), we say:

REPLACE the previous x -values by ...

Here are some examples that combine horizontal and vertical translations and scaling.

EXAMPLE:

State the transformations that take the graph of $y = f(x)$ to the graph of $y = 5f(3x - 1) + 4$.

Equation	Action	Graphical Result
$y = f(x)$	(starting place)	
$y = f(x - 1)$	replace every x by $x - 1$	move RIGHT 1
$y = f(3x - 1)$	replace every x by $3x$	horizontal shrink, $(a, b) \mapsto (\frac{a}{3}, b)$
$y = 5f(3x - 1)$	multiply the previous y -values by 5	vertical stretch, $(a, b) \mapsto (a, 5b)$
$y = 5f(3x - 1) + 4$	add 4 to the previous y -values	move UP 4

EXAMPLE:

State the transformations that take the graph of $y = \sqrt{x}$ to the graph of $y = 3\sqrt{\frac{x}{5} + 2} - 1$.

Equation	Action	Graphical Result
$y = \sqrt{x}$	(starting place)	
$y = 3\sqrt{x}$	multiply the previous y -values by 3	vertical stretch, $(a, b) \mapsto (a, 3b)$
$y = 3\sqrt{x + 2}$	replace every x by $x + 2$	move LEFT 2
$y = 3\sqrt{\frac{x}{5} + 2}$	replace every x by $\frac{x}{5}$	horizontal stretch, $(a, b) \mapsto (5a, b)$
$y = 3\sqrt{\frac{x}{5} + 2} - 1$	subtract 1 from the previous y -values	move DOWN 1