SATstuff#6

This week we'll continue working with problems involving algebra.

SOLVING EQUATIONS SIMULTANEOUSLY
The SAT likes to give you two equations involving two variables, and then ask you for the value of an expression. Try the problem below before you look at the two possible solutions:
If $3x - y = 1$ and $2x - 5y = 7$, then what is $x + 4y$?

SOLUTION #1 (traditional approach)
Use the substitution method:
– solve one of the equations for one of the variables (whichever is easier);
– then, substitute this information into the remaining equation
Solve the first equation for y: $y = 3x - 1$
Substitute into the second equation: $2x - 5(3x - 1) = 7$
Solve for x:
$2x - 15x + 5 = 7$
$-13x = 2$
$x = -\frac{2}{13}$
Then, find y: $y = 3x - 1 = 3(-\frac{2}{13}) - 1 = -\frac{19}{13}$
Finally, find the value of the desired expression:
$x + 4y = -\frac{2}{13} + 4(-\frac{19}{13}) = -\frac{78}{13} = -6$
Phew! Lots of work! Ugly fractions! Takes a long time!
You should know this approach, because it always works.
However, compare with the approach below:

SOLUTION #2 (shortcut)
Look for things that you can do with the two equations that brings the desired expression into the picture.
In this case, subtracting does the trick:

$$\begin{array}{rl} & 3x - y = 1 \\ - & [2x - 5y = 7] \\ \hline & x + 4y = -6 \end{array}$$

(subtract the left-hand sides; subtract the right-hand sides)

MUCH EASIER!

YOU TRY THESE:

(1) If $3x - y = 1$ and $2x - 5y = 7$, what is $5x - 6y$?

(2) If $3x - 5y = 2$ and $y + 2x = 6$, what is $x - 6y$?

(3) If $x^2 + xy = 3$ and $y^2 = xy + 1$, what is $x^2 + y^2$?

(1) 8
(2) -4
(3) 4

PRODUCTS and FACTORS

A *product* is an expression where the *last* operation is multiplication.
When you have a product, the quantities being multiplied are called the *factors*.

Clearly, xy is a product (since this represents x times y); the factors are x and y.

Also, $(x+2)(x-3)$ is a product. Here, the factors are $x+2$ and $x-3$.

PRODUCTS ARE PREFERRED in math, because of the following observation:

ZERO FACTOR LAW
For all real numbers x and y:

$$xy = 0 \quad \text{if and only if} \quad (x = 0 \text{ or } y = 0)$$

Partial translation:
The only way a product can equal zero is if one (or both) of its factors are equal to zero.

Think about this!
If I say to you: "I'm thinking of two numbers. When I multiply them together, I get zero."
What can you tell me about the numbers I'm thinking of?
At least one of them must be zero!

EXAMPLE:
Solve: $x^2(x-1)(2x+3) = 0$
DON'T MULTIPLY THINGS OUT!
Solution: $x = 0$ or $x = 1$ or $x = -\frac{3}{2}$

By the way, the ZERO on the right-hand side is critical!
If you were asked to solve the equation $x^2(x-1)(2x+3) = 1$,
life would be much more miserable.

YOU TRY THESE:

(4) Solve: $(3-x)(x^3)(5x-1) = 0$

(5) Solve: $(x^2+x)(1-x) = 0$

(6) Solve: $(2x-5)^3 = 0$

(7) Solve: Suppose that $x^2 = y^2$. What, if anything, can be said about x and y?

(4) $3, 0, \frac{1}{5}$ (5) $0, -1, 1$ (6) $\frac{5}{2}$ (7) $x = y$ or $x = -y$

FACTORING

Factoring is the process of taking a sum (things added) and renaming it as a product (things multiplied).

To factor, you often just use these familiar laws *backwards*:

$a(b+c) = ab + ac$ (the distributive law)

EXAMPLE:

$2t(x+1) + y(x+1) = (x+1)(2t+y)$

Locate the common factor(s);
write them down, using parentheses if there is more than one piece;
open up parentheses to hold what's left;
go back to each term and pick up the remaining parts.

$(x+a)(x+b) = x^2 + (a+b)x + ab$ (the basis for the FOIL method)

This is why you look for numbers that MULTIPLY to give you the last term,
and ADD to give you the number in the middle.

EXAMPLE: $x^2 + 5x + 4 = (x+1)(x+4)$

Here, we needed numbers that multiply to 4 and that add to 5: the numbers 1 and 4 work!

$(x-y)(x+y) = x^2 - y^2$ (the form $x^2 - y^2$ is called a DIFFERENCE OF SQUARES)

EXAMPLE: $(a+1)^2 - (b+2)^2 = ((a+1)-(b+2))((a+1)+(b+2)) = (a-b-1)(a+b+3)$

$(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$

$(x-y)^2 = (x-y)(x-y) = x^2 - 2xy + y^2$

If you see a quadratic expression (highest power is two)
involving two variables on the SAT,
then it will probably be either a difference of squares, or one of these!

YOU TRY THESE:

(8) FACTOR: $2x(x+3) - 6x^2(x+3)$

(9) FACTOR: $4x^2 - (x+3)^2$

(10) If $2x - y = 7$, then what is the value of $4x^2 - 4xy + y^2$?

(8) $2x(x+3)(1-3x)$

(9) $(x-3)(3x+3) = 3(x-3)(x+1)$

(10) 49

SOLVING QUADRATIC EQUATIONS

A *quadratic equation* involves only x^2, x, and constant (number) terms.

The key to solving them:

- write them in the form where you have ZERO on one side
- factor the other side
- use the ZERO FACTOR LAW

EXAMPLE:

If $x^2 + 4x = 5$, then what is x?

Answer:

$x^2 + 4x - 5 = 0$

$(x + 5)(x - 1) = 0$

$x = -5$ or $x = 1$

YOU TRY THESE:

(11) Solve: $x^2 = 6 - x$

(12) Solve: $x(x + 4) = 5$

(11) -3, 2 (12) -5, 1