This week we’ll review functions and properties of exponents.

**FUNCTIONS**

A FUNCTION is a rule that assigns to each input exactly one corresponding output. You can think of a function as ‘acting on’ an input and producing an output. Using normal function notation, if \( f \) is the name of the rule, and \( x \) is the input, then \( f(x) \) (pronounced as “\( f \) of \( x \)”) is the corresponding output.

![Diagram showing function notation]

So, what is \( g(3) \)? It is the output from the function \( g \), when the input is 3. What is \( f(x + h) \)? It is the output from the function \( f \), when the input is \( x + h \).

The SAT use more creative notation to illustrate the process of taking number(s), doing something to them, and getting a unique output.

For example, they might define \( \begin{vmatrix} x & y \end{vmatrix} \) to mean \( x + 3y \).

Then, \( \begin{vmatrix} 2 & 5 \end{vmatrix} \) represents \( 2 + 3 \times 5 = 17 \).

Verbalizing functions (rules) as SEQUENCES OF OPERATIONS

The best way to think of any function rule is as a sequence of operations.

For example, \( 2x + 3 \) represents the rule: take a number, multiply by 2, then add 3.

The rule \( 3x^2 \) represents the rule: take a number, square it, then multiply by 3.

The rule \( (3x)^2 \) represents the rule: take a number, multiply by 3, then square the result.

**TRY THESE:**

(a) In words, what does the function notation \( f(3) \) represent?

(b) Put the rule \( f(x) = 5x - 1 \) into words: take a number, ...

(c) Suppose that \( \begin{vmatrix} x & y \end{vmatrix} \) means mean \( 2x + y \).

Find \( \begin{vmatrix} 1 & 3 \end{vmatrix} \).

(d) Use a mathematical expression to represent this rule: take a number, multiply it by 5, then subtract 3.
If you’re shooting for a 600 on the Math SAT, then you should work at the pace of about one minute, thirty seconds per multiple-choice problem.

Get a feel for this pace by trying the following function problems:

1. For all positive numbers, $x$ represents the nearest even integer greater than $x$.
   If $x = 5$, then $\Rightarrow x$ is
   
   (A) 4   (B) 5   (C) 6   (D) 7   (E) 8

2. Let $x$ be any positive integer. The operation $* x$ is defined in the following way:
   $x^*$ represents the least prime number greater than $x$.
   If $x = 18$, then $x^* =$
   
   (A) 15   (B) 17   (C) 19   (D) 20   (E) 23

3. Let a “k-triple” be defined as $(\frac{k}{2}, k, \frac{3}{2}k)$ for some number $k$.
   Which of the following is a $k$-triple?
   
   (A) (0, 5, 10)   (B) (4\frac{1}{2}, 5, 6\frac{1}{2})   (C) (25, 50, 75)   (D) (250, 500, 1000)   (E) (450, 500, 650)

4. If $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ is defined to equal $wy - xz$, and $\begin{bmatrix} w & x \\ y & z \end{bmatrix} - K = 0$, then $K =$
   
   (A) $wy - wz$   (B) $xz + wy$   (C) $-xz$   (D) $xz - wy$   (E) $wy - xz$

The next two questions refer to the following definition:

$\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$ is a number square if $W + Z = X + Y$ and $2W = 3X$.

5. If $\begin{bmatrix} 3 & X \\ Y & 5 \end{bmatrix}$ is a number square, then what is the value of $Y$?
   
   (A) 0   (B) 2   (C) 4   (D) 6   (E) 8

6. If $\begin{bmatrix} W & X \\ Y & W \end{bmatrix}$ is a number square, then $Y =$
   
   (A) $\frac{3}{4}W$   (B) $W$   (C) $\frac{8}{3}W$   (D) $3W$   (E) $4W$

7. Let $\#$ be defined by $z \# w = z^w$.
   If $x = 5 \# a$, $y = 5 \# b$, and $a + b = 3$, then what is the value of $xy$?
   
   (A) 15   (B) 30   (C) 75   (D) 125   (E) 243
EXPONENT LAWS:

Exponent notation is a shorthand for repeated multiplication:

\[ x^3 \text{ means } x \cdot x \cdot x \]
\[ (a + b)^2 \text{ means } (a + b)(a + b) = a^2 + 2ab + b^2 \quad (\text{use FOIL}) \]

Here are the basic laws for working with exponents:

Things multiplied, same base, ADD the exponents:

\[ x^m x^n = x^{m+n} \]
Example: \((2^3)(2^5) = (2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) = 2^{3+5} = 2^8\)

Things divided, same base, SUBTRACT the exponents:

\[ \frac{x^m}{x^n} = x^{m-n} \]
Example: \[\frac{x^5}{x^3} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} = x^{5-3} = x^2\]

Something to a power, to a power, multiply the exponents:

\[(x^m)^n = x^{mn}\]
Example: \((x^2)^3 = (x^2)(x^2)(x^2) = (x \cdot x)(x \cdot x)(x \cdot x) = x^{2 \cdot 3} = x^6\)

Trade a negative exponent in for a "flip":

\[x^{-n} = \frac{1}{x^n} \text{ OR } \frac{1}{x^{-n}} = x^n\]
Example: \((a + b)^{-2} = \frac{1}{(a + b)^2}\)

Fractional exponents—the denominator tells the kind of root; the numerator is a power, which can go inside or outside:

\[x^{a/b} = (x^{1/b})^a = (\sqrt[b]{x})^a\] (usually, this name is easiest)
\[x^{a/b} = (x^a)^{1/b} = \sqrt[b]{x^a}\]
Example: \(8^{0/3} = (\sqrt[3]{8})^0 = 2^0 = 32\)

In particular, \(x^{1/2} = \sqrt{x}\).

Recall: \(\sqrt{x}\) is the nonnegative number which, when squared, gives \(x\):

Example: \(\sqrt{4} = 2\), even though both \(2^2 = 4\) and \((-2)^2 = 4\).

You can't take EVEN roots of NEGATIVE numbers: \(\sqrt{-4}\) is not defined.

SPECIAL CASES THAT YOU COME UP A LOT:
\[\sqrt{xy} = \sqrt{x}\sqrt{y}\] (only when \(x\) and \(y\) are BOTH positive)
\[\sqrt[\frac{\sqrt{x}}{\sqrt{y}}} = \frac{\sqrt{x}}{\sqrt{y}}\] (only when \(x\) and \(y\) are BOTH positive)

TRY THESE:
(a) \(\sqrt{(-2)(-2)} = \)
(b) \(\sqrt{\frac{9}{100}} = \)
(c) \(\sqrt{x^2} = \) (Be careful!)
(d) \((27)^{2/3} = \) (Do this WITHOUT a calculator.)

\[b = \varepsilon = \varepsilon \left( \frac{\text{LCM}}{\varepsilon} \right) = \varepsilon \left( \text{LCM} \right) \quad (p) \quad \text{or} \quad (q) \quad \varepsilon (\gamma) \]
Now, you try these:

1. If $2^y = 8$ and $y = \frac{x}{2}$, then $x =$
   (A) 6  (B) 5  (C) 4  (D) 3  (E) 2

2. If $x = 5^y$ and $y = z + 1$, then what is $\frac{x}{5}$ in terms of $z$?
   (A) $x$  (B) $z + 1$  (C) $5^x$  (D) $5^{z+1}$  (E) $5^{z+1}$

3. If $x + 1 = 7$, then $(x + 2)^2 =$
   (A) 25  (B) 36  (C) 49  (D) 64  (E) 81

4. If $\left(\frac{x + 1}{x}\right)^2 = 25$, then $\frac{1}{x^2} + x^2 =$
   (A) 23  (B) 24  (C) 25  (D) 27  (E) 624

5. If $(5^3)(2^5) = 4(10^k)$, then $k =$
   (A) 2  (B) 3  (C) 4  (D) 6  (E) 8

6. $\left[(2x^2y^3)^2\right]^3 =$
   (A) $4x^4y^6$  (B) $12x^4y^6$  (C) $64x^4y^6$  (D) $64x^{12}y^{18}$  (E) $64x^{64}y^{216}$

7. $a \cdot 3 \cdot b^2 \cdot \frac{1}{2} =$
   (A) $a^3b$  (B) $1.5ab^2$  (C) $1.5a^2b^2$  (D) $3ab$  (E) $6ab^2$
EXTRA PROBLEMS:

1. For any sentence \( J \), the expression \( N_t(J) \) is defined to mean the number of times the letter "t" appears in \( J \). If \( J \) is the sentence "All cats are good luck," then \( N_t(J) = \)
   (A) 0  (B) 1  (C) 2  (D) 3  (E) 4

Questions (2) and (3) refer to the following definition:
\(< x >\) is defined as 1 less than the number of digits in the integer \( x \).
For example, \(< 100 > = 3 - 1 = 2 \).

2. If \( x \) is a positive integer less than 1,000,001, then \( < x > \) is at most
   (A) 5  (B) 6  (C) 7  (D) 999,999  (E) 1,000,000

3. If \( x \) has 1,001 digits, then what is the value of \( <<< x >>> \)?
   (A) 997  (B) 1  (C) 0  (D) -1  (E) It cannot be determined from the information given.

4. For all numbers \( x, y, \) and \( z \), if the operation \( \phi \) is defined by the equation
   \( x \phi y = x + xy \), then \( x \phi \phi \phi z = \)
   (A) \( x + xy + xyz \)  (B) \( x + xyz \)  (C) \( x + xy + z + xz \)  (D) \( x + y + yz \)  (E) \( x + y + xyz \)

Questions (5) and (6) refer to the following definition:
\[ \frac{a \cdot b}{c} + \frac{b \cdot c}{a} + \frac{c \cdot a}{b} \] for all nonzero \( a, b, \) and \( c \).
For example,
\[ \frac{2 \cdot 4}{6} + \frac{4 \cdot 6}{2} + \frac{6 \cdot 2}{4} = \frac{4}{3} + 12 + 3 = 16 \frac{1}{3} \]

5. \[ \frac{3}{4} \]
   (A) 1  (B) 9  (C) 10  (D) 16  (E) 26

6. If \( x \neq 0 \), \[ \frac{x^2}{x^3} = \]
   (A) \( x^6 + x^4 + x^2 \)  (B) \( x^5 + x + \frac{1}{x} \)  (C) \( x^4 + x^3 + 1 \)  (D) \( x^4 + x^2 + 1 \)  (E) \( x^2 + x + 1 \)